

CLASSIFICATION OF HAMILTONIAN TORUS ACTIONS WITH TWO DIMENSIONAL QUOTIENTS

Yael Karshon and Susan Tolman

ABSTRACT. We construct all possible Hamiltonian torus actions for which all the non-empty reduced spaces are two dimensional (and not single points) and the manifold is connected and compact, or, more generally, the moment map is proper as a map to a convex set. This construction completes the classification of tall complexity one spaces.

CONTENTS

1. Introduction	1
2. Basic properties of complexity one spaces	13
Part I: Reconstruction	16
3. Topology of complexity one quotients	16
4. Lifting from the quotient	19
5. Gluing symplectic forms	22
6. Reconstruction	25
Part II: Classification	27
7. Compatibility of skeleton	27
8. Duistermaat-Heckman functions for tall complexity one models	29
9. Compatibility of the Duistermaat-Heckman function, and local existence	36
10. Proof of the existence theorems	43
References	44

1. INTRODUCTION

Fix a torus $T \cong (S^1)^{\dim T}$ with Lie algebra \mathfrak{t} and dual space \mathfrak{t}^* . Let T act on a symplectic manifold (M, ω) with moment map $\Phi: M \rightarrow \mathfrak{t}^*$,

2010 Mathematics Subject Classification. Primary 53D20, 53D35, 37J15. Secondary 57S15, 52B20, 14M25.

Yael Karshon is partially supported by an NSERC Discovery Grant. Susan Tolman is partially supported by NSF Grant DMS #07-07122.

so that

$$(1.1) \quad \iota(\xi_M)\omega = -d\langle \Phi, \xi \rangle \quad \text{for all } \xi \in \mathfrak{t},$$

where ξ_M is the vector field on M induced by ξ . Assume that the T -action is faithful (effective) on each connected component of M . We call (M, ω, Φ) a **Hamiltonian T-manifold**. An **isomorphism** between two Hamiltonian T -manifolds is an equivariant symplectomorphism that respects the moment maps. The **complexity** of (M, ω, Φ) is the difference $\frac{1}{2} \dim M - \dim T$; it is half the dimension of the reduced space $\Phi^{-1}(\alpha)/T$ at a regular value α in $\Phi(M)$. Assume that (M, ω, Φ) has complexity one; it is **tall** if every reduced space $\Phi^{-1}(\alpha)/T$ is a two dimensional topological manifold. If M is connected, \mathcal{T} is a convex open subset of \mathfrak{t}^* containing $\Phi(M)$, and the map $\Phi: M \rightarrow \mathcal{T}$ is proper, then we call $(M, \omega, \Phi, \mathcal{T})$ a **complexity one space**. For example, if M is compact and connected then $(M, \omega, \Phi, \mathfrak{t}^*)$ is a complexity one space, which it is tall exactly if the preimage of each vertex of the moment polytope $\Phi(M)$ is a fixed surface; see Corollary 2.4.

In this paper we complete our classification of tall complexity one spaces of arbitrary dimension. More precisely, in a previous paper [KT03] we defined an invariant of a tall complexity one space called the *painting*, which subsumes two other invariants: the *genus* and the *skeleton* (see page 4). We proved that these invariants, together with the Duistermaat-Heckman measure, determine the tall complexity one space up to isomorphism. In this paper we give a necessary and sufficient condition for a measure and a painting to arise from a tall complexity one space.

Symplectic toric manifolds (see page 4) serve as extremely important examples in symplectic geometry, illuminating many different aspects of the field. We hope that the existence theorems of this paper will enable complexity one spaces to serve a similar role. These spaces are more flexible than symplectic toric manifolds. For example, in a paper-in-progress, the second author uses the methods of [Tol98] to show that many complexity one spaces do not admit equivariant Kähler structures. Additionally, she constructs an infinite family of symplectic forms in a fixed cohomology class which are equivariantly deformation equivalent but are not equivariantly isotopic.

Symplectic toric manifolds are classified by their moment images [Del]; see [KL] for the non-compact case. Compact connected non-abelian complexity zero actions are determined by their moment image and the principal isotropy subgroup; this is the Delzant conjecture, recently proved in [Kn11] and [Los], following earlier work in [Igl, Woo].

The simplest complexity one spaces, compact connected symplectic 2-manifolds with no group action, are classified by their genus and total area [Mos]. Four dimensional compact connected complexity one spaces are classified in [Kar] (also see [AhHa, Au90, Au91]); see Example 1.7. Similar techniques apply to complexity one non-abelian group actions on six manifolds [Chi] and to two-torus actions on five dimensional K-contact manifolds [Noz]. From the algebraic geometric point of view, complexity one actions (of possibly nonabelian groups) have been studied in [KKMS, Chapter IV], [Tim96, Tim97], and [AlH, AlHS, AlP, Vol]. Moreover, a complexity one symplectic torus action on a compact symplectic manifold is Hamiltonian if and only if it has a fixed point [Kim].

Work on Hamiltonian circle actions on six manifolds, which have complexity two, appeared in [Li03, Li05, McD, Tol10, Gon, LiTo]. Finally, classification in arbitrary complexity is feasible for “centered spaces” ([Del, section 1], [KT05, §2], [KZ]).

We begin by recalling the invariants of complexity one spaces.

Let (M, ω, Φ) be a $2n$ dimensional Hamiltonian T -manifold. Recall that the Liouville measure on M is given by integrating the volume form $\omega^n/n!$ with respect to the symplectic orientation and that the **Duistermaat-Heckman measure** is the push-forward of the Liouville measure by the moment map. The isotropy representation at x is the linear representation of the stabilizer $\{\lambda \in T \mid \lambda \cdot x = x\}$ on the tangent space $T_x M$. Points in the same orbit have the same stabilizer, and their isotropy representations are linearly symplectically isomorphic; this isomorphism class is the **isotropy representation** of the orbit.

Now assume that (M, ω, Φ) is a tall complexity one Hamiltonian T -manifold. An orbit is **exceptional** if every nearby orbit in the same moment fiber $\Phi^{-1}(\alpha)$ has a strictly smaller stabilizer. Let M_{exc} denote the set of exceptional orbits in M/T , and let M'_{exc} denote the set of exceptional orbits of another tall complexity one space. An **isomorphism** from M_{exc} to M'_{exc} is a homeomorphism that respects the moment maps and sends each orbit to an orbit with the same (stabilizer and) isotropy representation. The **skeleton** of M is the set $M_{\text{exc}} \subset M/T$ with its induced topology, with each orbit labeled by its isotropy representation, and with the map $\bar{\Phi}: M_{\text{exc}} \rightarrow \mathfrak{t}^*$ that is induced from the moment map.

The next proposition is a slight modification of Proposition 2.2 of [KT03]; see Remark 1.9.

Proposition 1.2. *Let $(M, \omega, \Phi, \mathcal{T})$ be a tall complexity one space. There exists a closed oriented surface Σ and a map $f: M/T \rightarrow \Sigma$ so that*

$$(\overline{\Phi}, f): M/T \rightarrow (\text{image } \Phi) \times \Sigma$$

is a homeomorphism and the restriction $f: \Phi^{-1}(\alpha)/T \rightarrow \Sigma$ is orientation preserving for each $\alpha \in \text{image } \Phi$. Here, $\overline{\Phi}$ is induced by the moment map. Given two such maps f and f' , there exists an orientation preserving homeomorphism $\xi: \Sigma' \rightarrow \Sigma$ so that f is homotopic to $\xi \circ f'$ through maps which induce homeomorphisms $M/T \rightarrow (\text{image } \Phi) \times \Sigma$.

The **genus** of a tall complexity one space $(M, \omega, \Phi, \mathcal{T})$ is the genus of the surface $\Phi^{-1}(\alpha)/T$ for $\alpha \in \text{image } \Phi$. By Proposition 1.2, it is well defined. A **painting** is a map f from M_{exc} to a closed oriented surface Σ such that the map

$$(\overline{\Phi}, f): M_{\text{exc}} \rightarrow \mathcal{T} \times \Sigma$$

is one-to-one, where M_{exc} is the set of exceptional orbits. Two paintings, $f: M_{\text{exc}} \rightarrow \Sigma$ and $f': M'_{\text{exc}} \rightarrow \Sigma'$, are **equivalent** if there exist an isomorphism $i: M'_{\text{exc}} \rightarrow M_{\text{exc}}$ and an orientation preserving homeomorphism $\xi: \Sigma' \rightarrow \Sigma$ such that $f \circ i: M'_{\text{exc}} \rightarrow \Sigma$ and $\xi \circ f': M'_{\text{exc}} \rightarrow \Sigma$ are homotopic *through paintings*. Proposition 1.2 implies that there is a well-defined equivalence class of paintings associated to every tall complexity one space; just restrict f to M_{exc} .

Remark 1.3. The notion of a painting is simplest when $\overline{\Phi}: M_{\text{exc}} \rightarrow \mathfrak{t}^*$ is one-to-one, as in Examples 1.11 and 1.12. In this case, every map from M_{exc} to a closed oriented surface Σ is a painting, and two paintings $f: M_{\text{exc}} \rightarrow \Sigma$ and $f': M_{\text{exc}} \rightarrow \Sigma'$ are equivalent exactly if there exists an orientation preserving homeomorphism $\xi: \Sigma' \rightarrow \Sigma$ such that f and $\xi \circ f'$ are homotopic.

In the next two examples, we construct complexity one spaces out of **symplectic toric manifolds**, i.e., compact connected complexity zero Hamiltonian $(S^1)^n$ -manifolds. The moment image of a symplectic toric manifold is a Delzant polytope; see Remark 1.18. In fact, every Delzant polytope occurs as the moment image of a symplectic toric manifold, and this manifold is unique up to equivariant symplectomorphism [Del].

Example 1.4. Let (M, ω, ψ) be a symplectic toric manifold with moment image $\Delta = \psi(M) \subset \mathbb{R}^n$. The moment map induces a homeomorphism $\overline{\psi}: M/(S^1)^n \rightarrow \Delta$. Moreover, given $x \in \Delta$, let F_x be the smallest face of Δ containing x . The stabilizer of the preimage $\psi^{-1}(x)$ is the connected subgroup $H_x \subset (S^1)^n$ with Lie algebra

$$\mathfrak{h}_x = \{\xi \in \mathbb{R}^n \mid \langle \xi, y - z \rangle = 0 \text{ for all } y, z \in F_x\}.$$

Let $\Phi: M \rightarrow \mathbb{R}^{n-1}$ be the composition of the moment map ψ with the projection $\pi(x_1, \dots, x_n) = (x_1, \dots, x_{n-1})$. Then $(M, \omega, \Phi, \mathbb{R}^{n-1})$ is a complexity one space for the subtorus $(S^1)^{n-1} \subset (S^1)^n$. It is tall exactly if

$$(1.5) \quad \Delta_{\text{ceiling}} \cap \Delta_{\text{floor}} = \emptyset,$$

where

$$\begin{aligned} \Delta_{\text{ceiling}} &= \{x \in \Delta \mid x_n \geq x'_n \text{ for all } x' \in \pi^{-1}(\pi(x))\} \quad \text{and} \\ \Delta_{\text{floor}} &= \{x \in \Delta \mid x_n \leq x'_n \text{ for all } x' \in \pi^{-1}(\pi(x))\}. \end{aligned}$$

Assume that (1.5) holds.

For $x \in \Delta$ such that $\pi(x)$ is in the interior of $\pi(\Delta)$, the preimage $\psi^{-1}(x)$ is exceptional exactly if its $(S^1)^{n-1}$ stabilizer, $H_x \cap (S^1)^{n-1}$, is non-trivial. (This always occurs¹ if $\dim F_x < n - 1$.) The skeleton M_{exc} is the closure of the set of such orbits. The genus of (M, ω, Φ) is zero. The equivalence class of paintings associated to M includes the paintings that are constant on each component of M_{exc} . Finally, the Duistermaat-Heckman measure is the push-forward to \mathbb{R}^{n-1} of Lebesgue measure on Δ .

Example 1.6. Let $P \rightarrow \Sigma$ be a principal $(S^1)^n$ bundle over a closed oriented surface Σ of genus g with first Chern class $c_1(P) \in H^2(\Sigma, \mathbb{Z}^n)$, and let N be a symplectic toric manifold with moment map $\psi: N \rightarrow \mathbb{R}^n$. There exists a T -invariant symplectic form ω on $M := P \times_T N$ whose restriction to the fibers is the symplectic form on N ; the moment map $\Phi: M \rightarrow \mathbb{R}^n$ is given by $\Phi([p, n]) = \psi(n)$. See [GLS]. In this case, $(M, \omega, \Phi, \mathbb{R}^n)$ is a tall complexity space of genus g , $M_{\text{exc}} = \emptyset$, and the Duistermaat-Heckman measure of (M, ω, Φ) is Lebesgue measure on Δ times an affine function with slope $c_1(P)$.

Example 1.7. In [Kar], Karshon showed that a Hamiltonian circle action on a compact, connected symplectic four-manifold M is determined up to isomorphism by the following labelled graph: The vertices correspond to connected components of the fixed point set; each vertex is labelled by its moment map value, and – if the corresponding

¹ More generally, $H_x \cap (S^1)^{n-1}$ is trivial exactly if $\pi(\mathbb{Z}^n \cap TF_x) = \mathbb{Z}^{n-1}$, where $TF_x = \mathfrak{h}_x^\circ = \{\lambda(y - z) \mid \lambda \in \mathbb{R} \text{ and } y, z \in F_x\}$. To see this, note that $\pi(\mathbb{Z}^n \cap TF_x) = \mathbb{Z}^{n-1}$ exactly if every character of $(S^1)^{n-1}$ is the restriction of a character of $(S^1)^n$ that vanishes on H_x . If $H_x \cap (S^1)^{n-1}$ is not trivial, then there are characters of $(S^1)^{n-1}$ that don't vanish on $H_x \cap (S^1)^{n-1}$, and these can't be the restrictions of characters that vanish on H_x . On the other hand, if $H_x \cap (S^1)^{n-1} = \{1\}$, then either $H_x = \{1\}$ or $(S^1)^n \simeq H_x \times (S^1)^{n-1}$. In either case, every character of $(S^1)^{n-1}$ is the restriction of a character of $(S^1)^n$ that vanishes on H_x .

component is a two-dimensional fixed surface – the genus and area of that surface. The edges correspond to two spheres in M that the circle rotates at speed $k > 1$; such an edge is labelled by the integer k .

The space M is tall exactly if the minimum and maximum of the moment map are attained along two dimensional fixed surfaces; see Corollary 2.4. It is fairly straightforward to check that in this case the invariants that we describe in this paper determine, and are determined by, the labelled graph described above. In particular, the moment map identifies each component of the skeleton with an interval, and so every painting is trivial, i.e., equivalent to a painting that is locally constant.

By Theorem 1 of [KT03], the invariants that we have defined completely determine the tall complexity one space:

Theorem 1.8 (Global uniqueness). *Let $(M, \omega, \Phi, \mathcal{T})$ and $(M', \omega', \Phi', \mathcal{T})$ be tall complexity one spaces. They are isomorphic if and only if they have the same moment image² and Duistermaat-Heckman measure, the same genus, and equivalent paintings.*

Remark 1.9. In our definition of “equivalent paintings”, we require the homeomorphism ξ to be orientation preserving. This requirement, which is necessary for Theorem 1.8 to be true, was mistakenly omitted from [KT03, p.29]. Similarly, Definition 1.16 of the current paper is the correction to Definition 18.1 of [KT03]. Finally, both [KT03, Proposition 2.2] and its smooth analogue, [KT03, Lemma 18.4], should state that $f|_{\Phi^{-1}(\alpha)/T}$ and ξ are orientation preserving. The maps that are obtained in the proofs of these propositions in [KT03] do satisfy this additional requirement.

Before stating our most general existence theorem, Theorem 3, we give two existence theorems – Theorems 1 and 2 – that are easier to state and simpler to apply but are sufficient for constructing interesting examples. These two theorems are actually consequences of the most general theorem; all three are proved in Section 10; cf. Remark 1.10.

The simplest existence theorem. We now state our first existence theorem, which shows that – given a tall complexity one space – we can find another tall complexity one space with an isomorphic skeleton (and the same Duistermaat-Heckman measure) but a different genus and painting.

² Since the moment image is the support of the Duistermaat-Heckman measure, we could omit the condition that the spaces have the same moment image. Nevertheless, we will sometimes include this condition for emphasis.

Theorem 1. *Let $(M, \omega, \Phi, \mathcal{T})$ be a tall complexity one space. Let Σ be a closed oriented surface, and let $f: M_{\text{exc}} \rightarrow \Sigma$ be any painting. Then there exists a tall complexity one space $(M', \omega', \Phi', \mathcal{T})$ with the same moment image and Duistermaat-Heckman measure whose painting is equivalent to f .*

Remark 1.10. The special case of Theorem 1 where the genus of Σ is equal to the genus of M is easier to prove than the general case; see Theorem 6.1.

Example 1.11. Let (M, ω, ψ) be a six-dimensional symplectic toric manifold with moment image

$$\Delta = \{(x, y, z) \in [-3, 3] \times [-2, 2] \times [1, 4] \mid |x| \leq z \text{ and } |y| \leq z\}.$$

Let $\Phi: M \rightarrow \mathbb{R}^2$ be the composition of $\psi: M \rightarrow \mathbb{R}^3$ with the projection $(x, y, z) \mapsto (x, y)$. Then $(M, \omega, \Phi, \mathbb{R}^2)$ is a tall complexity one space, and Φ induces a homeomorphism from the skeleton M_{exc} to the set

$$\begin{aligned} \Phi(M_{\text{exc}}) = \{(x, y) \in \mathbb{R}^2 \mid |x| \leq |y| = 1, \text{ or } |y| \leq |x| = 1, \\ \text{or } 1 \leq |x| = |y| \leq 2\}; \end{aligned}$$

thus, M_{exc} is homotopy equivalent to S^1 . (See Example 1.4.)

Fix a closed oriented surface Σ . Let $[S^1, \Sigma]$ denote the set of homotopy classes of loops in Σ . By Remark 1.3, there is a one-to-one correspondence between equivalence classes of paintings $M_{\text{exc}} \rightarrow \Sigma$ and the quotient of $[S^1, \Sigma]$ by the action of the group $\text{Aut}(\Sigma)$ of orientation preserving homeomorphisms of Σ . If Σ has genus zero, then since Σ is simply connected any two paintings are equivalent. In contrast, if Σ has positive genus, then the quotient of $[S^1, \Sigma]$ by $\text{Aut}(\Sigma)$ is infinite. For example, if Σ has genus one, then this quotient is naturally isomorphic to the set of non-negative integers.

Therefore, if $g = 0$ then Theorem 1.8 implies that every tall complexity one space of genus g whose skeleton is isomorphic to M_{exc} and whose Duistermaat-Heckman measure is equal to that of (M, ω, Φ) is isomorphic to (M, ω, Φ) . In contrast, if $g > 0$ then Theorems 1.8 and 1 imply that there exist infinitely many non-isomorphic tall complexity one spaces with these properties.

Example 1.12. Fix an integer $n > 1$. Let (M, ω, ψ) be a $(2n + 4)$ -dimensional symplectic toric manifold with moment image

$$\begin{aligned} \Delta' = \{(x, y_1, \dots, y_n, z) \in [-3, 3] \times [-2, 2]^n \times [1, 4] \mid \\ |x| \leq z \text{ and } |y_i| \leq z \text{ for all } i = 1, \dots, n\}. \end{aligned}$$

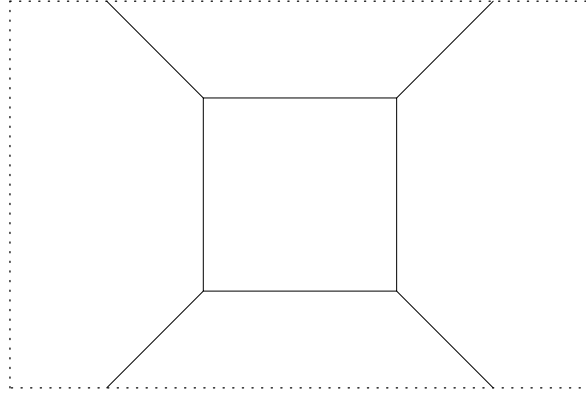


FIGURE 1. Moment image and skeleton for Example 1.11

Composing the moment map ψ with the projection $(x, y_1, \dots, y_n, z) \mapsto (x, y_1, \dots, y_n)$, we obtain a tall complexity one space $(M, \omega, \Phi, \mathbb{R}^{n+1})$ such that M_{exc} is homotopy equivalent to S^n and $\overline{\Phi}: M_{\text{exc}} \rightarrow \mathbb{R}^{n+1}$ is one-to-one. Moreover, the group of orientation preserving homeomorphisms acts trivially on the set $[S^n, \Sigma]$ of homotopy classes of maps from S^n to Σ if Σ is an oriented surface of genus 0, while $[S^n, \Sigma]$ itself is trivial if Σ has positive genus. Therefore, if $g > 0$ then Theorem 1.8 implies that every complexity one space of genus g whose skeleton is isomorphic to M_{exc} and whose Duistermaat-Heckman measure is equal to that of (M, ω, Φ) is isomorphic to (M, ω, Φ) . In contrast, if $g = 0$ then Theorems 1.8 and 1 give a bijection between the set of isomorphism classes of tall complexity one spaces with these properties and the set $[S^n, S^2]$. Thus, there are infinitely many non-isomorphic such spaces if $n = 2$ or $n = 3$.

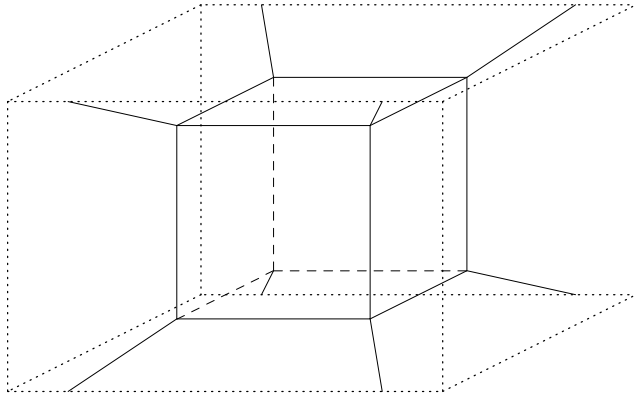


FIGURE 2. Moment image and skeleton for Example 1.12

The intermediate existence theorem. Our second existence theorem, Theorem 2, allows us to construct complexity one spaces with prescribed painting and moment image, even when the skeleton does not a-priori come from a complexity one space. To state this theorem, we need an abstract notion of “skeleton”. To apply this theorem, one must check that the skeleton and moment image satisfy certain conditions. These conditions are automatically satisfied whenever the the skeleton and moment image can be obtained from complexity one spaces over sufficiently small open sets in \mathfrak{t}^* ; see Lemma 7.4. This allows us to construct new examples by performing surgery that attaches pieces of *different* complexity one manifolds. Such surgeries were carried out in [Tol98, Mor]; this theorem gives a systematic way to perform such surgeries.

Definition 1.13. A **tall skeleton** over an open subset \mathcal{T} of \mathfrak{t}^* is a topological space S whose points are labeled by (equivalence classes of) representations of subgroups of T , together with a proper map $\pi: S \rightarrow \mathcal{T}$. This data must be locally modeled on the set of exceptional orbits of a tall complexity one space in the following sense. For each point $s \in S$, there exists a tall complexity one Hamiltonian T -manifold (M, ω, Φ) with exceptional orbits $M_{\text{exc}} \subset M/T$, and a homeomorphism Ψ from a neighbourhood of s to an open subset of M_{exc} that respects the labels and such that $\bar{\Phi} \circ \Psi = \pi$, where $\bar{\Phi}: M_{\text{exc}} \rightarrow \mathfrak{t}^*$ is induced from the moment map. An **isomorphism** between tall skeletons (S', π') and (S, π) is a homeomorphism $i: S' \rightarrow S$ that sends each point to a point with the same isotropy representation and such that $\pi' = \pi \circ i$; cf. [KT03, p. 72].

Remark 1.14. In [KT03, Definition 16.1] we called this notion “skeleton”. Here we added the adjective “tall” in order to later allow for skeletons that are not tall.

Example 1.15. If $(M, \omega, \Phi, \mathcal{T})$ is a tall complexity one space, the set M_{exc} , labeled with the isotropy representations, together with the map $\bar{\Phi}$ that is induced by the moment map, is a tall skeleton over \mathcal{T} ; see Lemma 7.2.

Definition 1.16. Let (S, π) be a tall skeleton over an open set $\mathcal{T} \subset \mathfrak{t}^*$ and let Σ be a closed oriented surface. A **painting** is a map $f: S \rightarrow \Sigma$ such that the map $(\pi, f): S \rightarrow \mathcal{T} \times \Sigma$ is one-to-one. Paintings $f: S \rightarrow \Sigma$ and $f': S' \rightarrow \Sigma'$ are **equivalent** if there exists an isomorphism $i: S' \rightarrow S$ and an orientation preserving homeomorphism $\xi: \Sigma' \rightarrow \Sigma$ such that $f \circ i: S' \rightarrow \Sigma$ and $\xi \circ f': S' \rightarrow \Sigma$ are homotopic through paintings.

The notions of painting and of equivalence of paintings given in Definition 1.16 are consistent with our earlier definitions, which only applied to the special case $(S, \pi) = (M_{\text{exc}}, \overline{\Phi})$.

Let ℓ denote the integral lattice in \mathfrak{t} and ℓ^* the weight lattice in \mathfrak{t}^* . Thus, $\ell = \ker(\exp: \mathfrak{t} \rightarrow T)$ and $\ell^* \cong \text{Hom}(T, S^1)$. Here, the Lie algebra of S^1 is identified with \mathbb{R} by setting the exponential map $\mathbb{R} \rightarrow S^1$ to be $t \mapsto e^{2\pi it}$. Let \mathbb{R}_+ denote the set of non-negative numbers.

Definition 1.17. A subset $C \subset \mathfrak{t}^*$ is a **Delzant cone**³ at $\alpha \in \mathfrak{t}^*$ if there exist an integer $0 \leq k \leq n$ and a linear isomorphism $A: \mathbb{R}^n \rightarrow \mathfrak{t}^*$ that sends \mathbb{Z}^n onto the weight lattice ℓ^* , such that

$$C = \alpha + A(\mathbb{R}_+^k \times \mathbb{R}^{n-k}).$$

Let \mathcal{T} be an open subset of \mathfrak{t}^* . A subset $\Delta \subset \mathcal{T}$ is a **Delzant subset** if it is closed in \mathcal{T} and if for every point $\alpha \in \Delta$ there exist a neighbourhood $U \subset \mathcal{T}$ and a Delzant cone C at α such that $\Delta \cap U = C \cap U$.

Remark 1.18. A compact convex set $\Delta \subset \mathfrak{t}^*$ is a Delzant subset exactly if it is a **Delzant polytope**, i.e., a convex polytope such that at each vertex the edge vectors are generated by a basis to the lattice.

Remark 1.19. If Δ is a Delzant subset of a *convex* open subset $\mathcal{T} \subset \mathfrak{t}^*$ then, by the Tietze-Nakajima theorem [Tie, Nak], Δ is convex exactly if it is connected; see [BjKa].

Definition 1.20. The **moment cone** corresponding to a point s in a tall skeleton (S, π) is the cone

$$C_s := \pi(s) + (i_H^*)^{-1}(\text{image } \Phi_s) \text{ in } \mathfrak{t}^*,$$

where the label associated to s is a linear symplectic representation of the subgroup H of T with quadratic moment map Φ_s , and where $i_H^*: \mathfrak{t}^* \rightarrow \mathfrak{h}^*$ is the natural projection map. It is straightforward to check that C_s is the moment image of the complexity one model corresponding to s ; see Definition 1.22.

Definition 1.21. Let \mathcal{T} be an open subset of \mathfrak{t}^* . A Delzant subset Δ of \mathcal{T} and a tall skeleton (S, π) over \mathcal{T} are **compatible** if for every point $s \in S$ there exists a neighbourhood U of $\pi(s)$ in \mathcal{T} such that $U \cap \Delta = U \cap C_s$, where C_s is the moment cone corresponding to s .

Let $(M, \omega, \Phi, \mathcal{T})$ be a tall complexity one space. Then its moment map image is a convex Delzant subset of \mathcal{T} that is compatible with the skeleton $(M_{\text{exc}}, \overline{\Phi})$; see Lemma 7.4. Our next theorem shows that this compatibility condition is also sufficient for a subset of \mathcal{T} and a painting to arise from a complexity one space.

³ Such a set is also called a “unimodular cone”.

Theorem 2. *Let (S, π) be a tall skeleton over a convex open subset $\mathcal{T} \subset \mathfrak{t}^*$. Let $\Delta \subset \mathcal{T}$ be a convex Delzant subset that is compatible with (S, π) . Let Σ be a closed oriented surface, and let $f: S \rightarrow \Sigma$ be a painting. Then there exists a tall complexity one space $(M, \omega, \Phi, \mathcal{T})$ with moment map image Δ whose associated painting is equivalent to f .*

The most general existence theorem. Our final existence theorem, Theorem 3, provides a complete list of all the possible values of the invariants of tall complexity one spaces. Together with Theorem 1.8, this gives a complete classification of tall complexity one spaces.

The **Duistermaat-Heckman function** of a Hamiltonian T -manifold is a real valued function on the moment image whose product with Lebesgue measure is equal to the Duistermaat-Heckman measure. If such a function exists, then it is almost unique; any two such functions are equal almost everywhere. When we say that the Duistermaat-Heckman function of a Hamiltonian T -manifold has some property (e.g., continuity), we mean that this holds after possibly changing the function on a set of measure zero. Here, we normalize Lebesgue measure on \mathfrak{t}^* such that the volume of the quotient \mathfrak{t}^*/ℓ^* is one.

A function $\rho: \mathfrak{t}^* \rightarrow \mathbb{R}$ is **integral affine** if it has the form

$$\rho(x) = \langle x, A \rangle + B,$$

where A is an element of the integral lattice $\ell \subset \mathfrak{t}$, where $B \in \mathbb{R}$, and where $\langle \cdot, \cdot \rangle$ is the pairing between \mathfrak{t}^* and \mathfrak{t} . The Duistermaat-Heckman theorem implies that the Duistermaat-Heckman function of a complexity one space with no exceptional orbits is integral affine.

Once and for all, fix an inner product on \mathfrak{t} . Let a closed subgroup $H \subset T$ act on \mathbb{C}^n as a subgroup of $(S^1)^n$ with quadratic moment map $\Phi_H: \mathbb{C}^n \rightarrow \mathfrak{h}^*$. Let $\mathfrak{h}^0 \subset \mathfrak{t}^*$ be the annihilator of the Lie algebra \mathfrak{h} , and consider the model

$$Y = T \times_H \mathbb{C}^n \times \mathfrak{h}^0,$$

where $[ta, z, \nu] = [t, az, \nu]$ for all $(t, z, \nu) \in T \times \mathbb{C}^n \times \mathfrak{h}^0$ and $a \in H$. There exists a T invariant symplectic form on Y with moment map

$$\Phi_Y([t, z, \nu]) = \alpha + \Phi_H(z) + \nu,$$

where $\alpha \in \mathfrak{t}^*$ and where we use the inner product to embed \mathfrak{h}^* in \mathfrak{t}^* . The isotropy representation of the orbit $\{[t, 0, 0]\}$ determines the model up to permutation of the coordinates in \mathbb{C}^n . If $\dim T = \frac{1}{2} \dim Y - 1$, or, equivalently, $n = h + 1$ where $h = \dim H$, we call the space Y a **complexity one model**.

Definition 1.22. Given a point s in a tall skeleton S , the **corresponding model** is the model $Y = T \times_H \mathbb{C}^{h+1} \times \mathfrak{h}^0$ such that s is labeled by the isotropy representation of $\{[t, 0, 0]\}$ in Y .

Such a model exists and is unique up to permutation of the coordinates in \mathbb{C}^{h+1} . Moreover, by Corollary 2.5, the corresponding model is always tall.

Let Y be a tall complexity one model. In Section 8 we define the *Duistermaat-Heckman functions for truncations of the model*. (In fact, such functions are the Duistermaat-Heckman functions of compact spaces that are obtained from Y by extending the action to a toric action, choosing a subcircle that is complementary to the original action, and taking a symplectic cut with respect to this circle.)

Definition 1.23. Let (S, π) be a tall skeleton over an open subset \mathcal{T} of \mathfrak{t}^* . Let $\Delta \subset \mathcal{T}$ be a convex Delzant subset that is compatible with (S, π) . Fix a point $\alpha \in \Delta$. A function $\rho: \Delta \rightarrow \mathbb{R}_{>0}$ is **compatible with (S, π) at the point $\alpha \in \Delta$** if there exist for each $s \in \pi^{-1}(\alpha)$ a Duistermaat-Heckman function ρ_s for a truncation of the tall complexity one model associated to s such that the difference

$$(1.24) \quad \rho - \sum_{s \in \pi^{-1}(\alpha)} \rho_s$$

is integral affine on some neighbourhood of α in Δ . (In particular, if $\pi^{-1}(\alpha)$ is empty, then the condition is that ρ itself be integral affine near α .) The function ρ is **compatible with (S, π)** if it is compatible with (S, π) at every $\alpha \in \Delta$.

Remark 1.25. The above notion of “compatible” is in fact well defined; moreover, the difference between any two compatible functions is integral affine near α . To see this, let (S, π) be a tall skeleton over \mathcal{T} ; fix $\alpha \in \mathcal{T}$. The preimage $\pi^{-1}(\alpha) \subset S$ is finite; see Corollary 2.6. Thus, the summation in (1.24) is finite. By Corollary 8.22, for each $s \in \pi^{-1}(\alpha)$, there exists a Duistermaat-Heckman function ρ_s for a truncation of the tall complexity one model Y_s associated to s ; moreover, ρ_s is defined on a neighborhood of α in image Φ_{Y_s} . By Definitions 1.20 and 1.21, the moment cone $C_s = \text{image } \Phi_{Y_s}$ coincides with Δ near α for all $s \in \pi^{-1}(\alpha)$. Thus, the function in (1.24) is defined on a neighbourhood of α in Δ . Finally, if both ρ_s and ρ'_s are Duistermaat-Heckman functions for truncations of the model Y_s , then by Corollary 8.23 there exists a neighbourhood of α in C_s , hence in Δ , on which the difference $\rho_s - \rho'_s$ coincides with an integral affine function.

The Duistermaat-Heckman function of a tall complexity one space is compatible with the skeleton; see Proposition 9.2. Our final theorem shows that this compatibility condition is also sufficient for a function, a subset of \mathcal{T} , and a painting to arise from a complexity one space.

Theorem 3 (Global existence). *Let (S, π) be a tall skeleton over a convex open subset $\mathcal{T} \subset \mathfrak{t}^*$, let $\Delta \subset \mathcal{T}$ be a convex Delzant subset that is compatible with (S, π) , and let $\rho: \Delta \rightarrow \mathbb{R}_{>0}$ be a function that is compatible with (S, π) . Let Σ be a closed oriented surface, and let $f: S \rightarrow \Sigma$ be a painting. Then there exists a tall complexity one space over \mathcal{T} with moment image Δ and Duistermaat-Heckman function ρ whose painting is equivalent to f .*

Section 2 contains some general facts about complexity one spaces. The remainder of the paper is divided into two parts. Sections 3 through 6 constitute Part I of the paper and lead to Theorem 6.1. This is a *reconstruction theorem* in the sense that we take a tall complexity one space, break it into pieces, and glue the pieces together so as to obtain a new complexity one space. In Section 3, we prove some facts about the cohomology of spaces that are locally modeled on the quotients of complexity one spaces. In Section 4, we glue local pieces of complexity one spaces as T -manifolds. In section 5, we show how to arrange that the symplectic forms on these local pieces will agree on their overlaps. In Section 6, we use the technology developed so far and a crucial technical result from our previous paper [KT03, Prop. 20.1] to prove Theorem 6.1. Sections 7 through 10 constitute Part II of the paper. In Section 7 we show that the moment map image and skeleton of a tall complexity one space satisfy our compatibility conditions. In Section 9 we use technical results from Section 8 to show that the Duistermaat-Heckman measure of a complexity one space is compatible with its skeleton, and we give a local existence theorem: any compatible data *locally* comes from a complexity one space. Finally, in Section 10, we combine these results with a variant of the reconstruction theorem from Section 6 to prove the main existence theorems: Theorems 1, 2, and 3.

2. BASIC PROPERTIES OF COMPLEXITY ONE SPACES

In this section we recall the local normal form theorem and the convexity package, and analyze some of their basic consequences for complexity one Hamiltonian torus actions.

Local normal form theorem.

For every orbit x in a Hamiltonian T -manifold M there is a corresponding model $Y = T \times_H \mathbb{C}^n \times \mathfrak{h}^0$ such that the isotropy representation of the orbit $\{[t, 0, 0]\}$ is the same as that of x . The **local normal form for Hamiltonian torus actions** asserts that there exists an equivariant symplectomorphism from an invariant neighbourhood of x in M to an invariant open subset of Y that carries x to $\{[t, 0, 0]\}$; see [GS84, Mar].

Convexity package.

Let (M, ω, Φ) be a connected Hamiltonian T -manifold. Suppose that there exists a convex open subset \mathcal{T} of \mathfrak{t}^* that contains $\Phi(M)$ and such that $\Phi: M \rightarrow \mathcal{T}$ is proper. Then we have the following *convexity package*.

Convexity: The moment map image, $\Phi(M)$, is convex.

Connectedness: The moment fiber, $\Phi^{-1}(\alpha)$, is connected for all $\alpha \in \mathcal{T}$.

Stability: As a map to $\Phi(M)$, the moment map is open.

These three properties also hold for the moment map of a local model. Note that together the three properties imply that the moment map preimage of every convex set is connected. Moreover, by convexity, stability, and the local normal form theorem, $\Delta := \Phi(M)$ is a convex polyhedral subset of \mathcal{T} whose faces have rational slopes.

For the compact case, see [Ati], [GS82], and [Sja, Theorem 6.5]; also see [LeTo]. For convexity and connectedness in the case of proper moment maps to open convex sets, see [LMTW]. Stability then follows from the local normal form theorem and stability for local models; see [Sja, Theorem 5.4 and Example 5.5]. Also see [BjKa, section 7].

Remark 2.1. In the situation described above, the set of α in Δ such that the reduced space $\Phi^{-1}(\alpha)/T$ is a single point is a union of closed faces of Δ . To see this, fix a point $x \in \Delta$ and let F_x be the smallest (closed) face containing x . The preimage $M_{F_x} = \Phi^{-1}(F_x)$ is a symplectic manifold with a Hamiltonian T action. (This follows from the local normal form theorem, which we will henceforth use without comment.) Moreover, since F_x is convex, M_{F_x} is connected. By stability, the subgroup that acts trivially on M_{F_x} has Lie algebra $(TF_x)^\circ = \{\xi \in \mathfrak{t} \mid \langle (y - z), \xi \rangle = 0 \text{ for all } y, z \in F_x\}$. Moreover, if we assume that $\Phi^{-1}(x)$ is a single orbit, then stability implies that $\dim M_{F_x} = 2 \dim F_x$. Because the moment map level sets of M are connected, this implies that the level sets over F_x are single orbits, as required.

Some consequences.

In order to apply these theorems to complexity one spaces, we now analyze complexity one models.

Lemma 2.2. *Let $Y = T \times_H \mathbb{C}^{h+1} \times \mathfrak{h}^0$ be a complexity one model with moment map $\Phi_Y([t, z, \nu]) = \alpha + \Phi_H(z) + \nu$.*

- *If the moment map Φ_Y is proper, then the level set $\Phi_Y^{-1}(\alpha)$ consists of a single orbit.*
- *If the moment map Φ_Y is not proper, then there exists a homeomorphism*

$$(2.3) \quad Y/T \rightarrow (\text{image } \Phi_Y) \times \mathbb{C}$$

whose first component is induced from the moment map and whose second component takes the set of exceptional orbits to zero.

Moreover, the map (2.3) carries the symplectic orientation of the smooth part of each reduced space to the complex orientation of \mathbb{C} .

Proof. The first assertion follows from the formula for Φ_Y and the fact that Φ_H is quadratic, hence homogeneous. For the second assertion, see [KT01, Lemma 6.2], and see [KT01, Definition 8.2] and the sentence that follows it. \square

Corollary 2.4 (Short/tall dichotomy). *Let $(M, \omega, \Phi, \mathcal{T})$ be a complexity one space with moment image $\Delta = \text{image } \Phi$. A nonempty reduced space $\Phi^{-1}(\alpha)/T$ is either a connected two dimensional oriented topological manifold or a single point. The set of α where the latter occurs is a union of closed faces of Δ .*

Proof. By Lemma 2.2 and the local normal form theorem there exists an open set $U \subset M/T$ such that, for each $\alpha \in \Delta$, the intersection $\Phi^{-1}(\alpha)/T \cap U$ is a two dimensional oriented topological manifold, and its complement in $\Phi^{-1}(\alpha)/T$ is discrete. Hence, by the connectedness of the level sets, every non-empty reduced space either consists of a single point or is a connected two dimensional oriented topological manifold. The last claim follows from Remark 2.1. \square

By Corollary 2.4, we can use the following fact to understand Δ_{tall} .

Corollary 2.5. *In a tall complexity one Hamiltonian T -manifold, the corresponding local models are tall.*

Proof. By Lemma 2.2, a complexity one model Y is tall exactly if there exists a neighbourhood of $\{[t, 0, 0]\}$ in Y that is tall. Hence, the claim follows immediately from the local normal form theorem. \square

Corollary 2.6. *Let (S, π) be a tall skeleton over an open subset \mathcal{T} of \mathfrak{t}^* . Then $\pi^{-1}(\alpha)$ is finite for every $\alpha \in \mathcal{T}$.*

Proof. Let s be a point in S and let $Y = T \times_H \mathbb{C}^{h+1} \times \mathfrak{h}^0$ be the corresponding complexity one model, which is tall by Corollary 2.5. By the local normal form theorem and Lemma 2.2 there exists a neighbourhood of s in S whose intersection with $\pi^{-1}(\alpha)$ consists of the single element set $\{s\}$. The result then follows from the properness of π . \square

PART I: RECONSTRUCTION

3. TOPOLOGY OF COMPLEXITY ONE QUOTIENTS

In this section, we prove two results about the topology of complexity one quotients which we will need in order to prove the main propositions in Sections 4 and 5. For future reference, whenever possible we will allow complexity one spaces that are not tall.

For a topological space X and a presheaf \mathcal{S} of abelian groups on X , we let $\check{H}^i(X, \mathcal{S})$ denote the Čech cohomology of \mathcal{S} . If X is paracompact⁴, this agrees with the Čech cohomology $\check{H}^i(X, \mathcal{S}^+)$ of the sheafification \mathcal{S}^+ of \mathcal{S} and with the sheaf cohomology $H^i(X, \mathcal{S}^+)$ of \mathcal{S}^+ that is defined through derived functors. Voir Théorème 5.10.1 et le Corollaire de Théorème 5.10.2 de [God, chapitre II].

Consider a continuous map of topological spaces, $\overline{\Phi}: Q \rightarrow B$. Given an abelian group A and a non-negative integer i , define a presheaf \mathcal{H}_A^i on B by

$$\mathcal{H}_A^i(U) = \check{H}^i(\overline{\Phi}^{-1}(U); A) \quad \text{for each open set } U \subset B.$$

Note that $\mathcal{H}_A^i(\emptyset) = \{0\}$. This presheaf is the push-forward by $\overline{\Phi}: Q \rightarrow B$ of the presheaf on Q that associates to each open set $W \subset Q$ the group $\check{H}^i(W; A)$. In general, neither presheaf is a sheaf.

Proposition 3.1. *Let Q be a topological space, \mathcal{T} be an open subset of \mathfrak{t}^* , and $\overline{\Phi}: Q \rightarrow \mathcal{T}$ be a continuous map such that $\Delta = \text{image } \overline{\Phi}$ is convex. Assume that for every point in \mathcal{T} there exists a convex neighbourhood U in \mathcal{T} , a complexity one space $(M_U, \omega_U, \Phi_U, U)$, and a homeomorphism from $\overline{\Phi}^{-1}(U)$ to M_U/T that carries $\overline{\Phi}|_{\overline{\Phi}^{-1}(U)}$ to the map $\overline{\Phi}_U: M_U/T \rightarrow U$ induced by Φ_U . Then for any abelian group A ,*

$$\check{H}^i(\mathcal{T}, \mathcal{H}_A^0) = \check{H}^i(\mathcal{T}, \mathcal{H}_A^1) = 0 \quad \text{for all } i > 0.$$

⁴We adopt the convention that, by definition, every paracompact space is Hausdorff.

Moreover, if at least one of the spaces M_U is not tall, then

$$\check{H}^0(\mathcal{T}, \mathcal{H}_A^2) = 0.$$

Proof. We first show that Q is paracompact. Let \mathfrak{W} be an arbitrary open covering of Q . There exists a locally finite covering ν of \mathcal{T} by open balls such that every $B \in \nu$, the preimage $\overline{\Phi}^{-1}(\overline{B})$ of the closure \overline{B} of B is compact. For each $B \in \nu$, let $\mathfrak{W}_B \subset \mathfrak{W}$ be a finite subset that covers $\overline{\Phi}^{-1}(\overline{B})$; then

$$\bigcup_{B \in \nu} \{W \cap f^{-1}(B) \mid W \in \mathfrak{W}_B\}$$

is a locally finite open refinement of \mathfrak{W} that covers Q .

The map $\overline{\Phi}: Q \rightarrow \Delta \subset \mathcal{T}$ induces presheaves \mathcal{H}_A^j on Δ and \mathcal{T} . Moreover, since $\mathcal{H}_A^j(U) = \mathcal{H}_A^j(U \cap \Delta)$ for all open $U \subset \mathcal{T}$, we have

$$\check{H}^i(\mathcal{T}, \mathcal{H}_A^j) = \check{H}^i(\Delta, \mathcal{H}_A^j) \quad \text{for all } i \text{ and } j.$$

Let $(\mathcal{H}_A^j)^+$ denote the sheafification of the presheaf \mathcal{H}_A^j on Δ . Because the Čech cohomology of a presheaf on a paracompact space is equal to that of its sheafification, it is enough to prove that

$$\check{H}^i(\Delta, (\mathcal{H}_A^0)^+) = \check{H}^i(\Delta, (\mathcal{H}_A^1)^+) = 0 \quad \text{for all } i > 0$$

and that, if at least one of the spaces M_U is not tall, then

$$\check{H}^0(\Delta, (\mathcal{H}_A^2)^+) = 0.$$

Assume first that all of the complexity one spaces M_U are tall. By Proposition 1.2, this implies that for every point in \mathcal{T} there exists a convex neighbourhood U in \mathcal{T} , a surface Σ , and a function $f: \overline{\Phi}^{-1}(U) \rightarrow \Sigma$, such that

$$(\overline{\Phi}, f): \overline{\Phi}^{-1}(U) \rightarrow (\Delta \cap U) \times \Sigma$$

is a homeomorphism. Hence, $(\mathcal{H}_A^0)^+$ is a constant sheaf and $(\mathcal{H}_A^j)^+$ is a locally constant sheaf for all $j > 0$. Since Δ is convex, it is contractible; thus $\check{H}^i(\Delta, (\mathcal{H}_A^j)^+) = \{0\}$ for all j and all $i > 0$.

Next, assume that at least one of the complexity one spaces M_U is not tall. Let Δ_{tall} denote the set of $\alpha \in \Delta$ such that $\overline{\Phi}^{-1}(\alpha)$ is a connected two dimensional oriented topological manifold; let $\Delta_{\text{short}} = \Delta \setminus \Delta_{\text{tall}}$. Corollary 2.4 implies that Δ_{tall} is open in Δ and $\overline{\Phi}^{-1}(\alpha)$ is a single point for all $\alpha \in \Delta_{\text{short}}$.

By assumption, for every point in \mathcal{T} there exists a convex neighbourhood U , a complexity one space $(M_U, \omega_U, \Phi_U, U)$, and a homeomorphism from $\overline{\Phi}^{-1}(U)$ to M_U/T that carries $\overline{\Phi}|_{\overline{\Phi}^{-1}(U)}$ to the map

$\overline{\Phi}_U: M_U/T \rightarrow U$ induced by Φ_U . In fact, the convexity package implies that the preimage $\Phi_U^{-1}(V)$ is connected for any convex subset $V \subset U$; see page 14. Hence, the neighbourhood U can be chosen to be arbitrarily small.

In particular, every $\alpha \in \Delta$ has arbitrarily small neighbourhoods whose pre-images in Q are connected. Hence, $(\mathcal{H}_A^0)^+$ is a constant sheaf. Since Δ is convex, this implies that $\check{H}^i(\Delta, (\mathcal{H}_A^0)^+) = 0$ for all $i > 0$.

The following result is proved in [KT01, Lemma 5.7]:

(3.2) Let (M, ω, Φ, U) be a complexity one space. Suppose that $\Phi^{-1}(\alpha)$ consists of a single orbit. Then every neighbourhood of α contains a smaller neighbourhood V such that the quotient $\Phi^{-1}(V)/T$ is contractible. Moreover, every regular non-empty symplectic quotient $\Phi^{-1}(y)/T$ in $\Phi^{-1}(V)/T$ is homeomorphic to a 2-sphere.

By Proposition 1.2, the genus of the reduced space is locally constant on Δ_{tall} . Hence, since regular values are dense, (3.2) implies that this genus is zero for all two dimensional reduced spaces over a neighbourhood of Δ_{short} . Since Δ is connected and Δ_{short} is not empty, this implies that every two dimensional reduced space has genus zero. Hence, by Proposition 1.2 and (3.2), $(\mathcal{H}_A^1)^+$ is the zero sheaf. Therefore, $\check{H}^i(\Delta, (\mathcal{H}_A^1)^+) = 0$ for all $i > 0$.

Finally, consider a global section $\gamma \in \check{H}^0(\Delta, (\mathcal{H}_A^2)^+)$. By (3.2), the support of γ is a subset of Δ_{tall} . Therefore, since the restriction of $(\mathcal{H}_A^2)^+$ to Δ_{tall} is a locally constant sheaf by Proposition 1.2, the support of γ is an open and closed subset of Δ_{tall} . Since Δ is connected and Δ_{short} is non-empty, this implies that $\gamma = 0$. Thus, $\check{H}^0(\Delta, (\mathcal{H}_A^2)^+) = 0$. \square

Proposition 3.3. *Let $(M, \omega, \Phi, \mathcal{T})$ be a complexity one space. The restriction map $H^2(M/T; \mathbb{Z}) \rightarrow H^2(\Phi^{-1}(y)/T; \mathbb{Z})$ is one-to-one for each $y \in \text{image } \Phi$.*

Proof. If the complexity one space is tall, this proposition is an immediate consequence of Proposition 1.2. So assume that it is not tall. Let $\overline{\Phi}: M/T \rightarrow \mathcal{T}$ be the map induced by Φ . Then there is the Leray spectral sequence converging to $H^*(M/T; \mathbb{Z})$ with

$$E_2^{i,j} = \check{H}^i(\mathcal{T}, \mathcal{H}_{\mathbb{Z}}^j);$$

see [God, chap. II, Thm. 4.17.1]. By Proposition 3.1, $E_2^{i,j} = 0$ for all i and j such that $i + j = 2$. Consequently, $H^2(M/T; \mathbb{Z}) = 0$. \square

4. LIFTING FROM THE QUOTIENT

An important step in gluing together local pieces of complexity one spaces is to glue them together as T -manifolds. To carry this out, which we will do in this section, we need a notion of diffeomorphisms of quotient spaces.

Let a compact torus T act on a manifold N . The quotient N/T can be given a natural differential structure, consisting of the sheaf of real-valued functions whose pullbacks to N are smooth.⁵ We say that a map $h: N/T \rightarrow N'/T$ is **smooth** if it pulls back smooth functions to smooth functions; it is a **diffeomorphism** if it is smooth and has a smooth inverse. If N and N' are oriented, the choice of an orientation on T determines orientations on the smooth part of N/T and N'/T . Whether or not a diffeomorphism $f: N/T \rightarrow N'/T$ preserves orientation is independent of this choice.

We now recall several definitions from [KT01].

Definition 4.1. Let a torus T act on oriented manifolds M and M' with T -invariant maps $\Phi: M \rightarrow \mathfrak{t}^*$ and $\Phi': M' \rightarrow \mathfrak{t}^*$. A **Φ - T -diffeomorphism** from (M, Φ) to (M', Φ') is an orientation preserving equivariant diffeomorphism $f: M \rightarrow M'$ that satisfies $\Phi' \circ f = \Phi$.

Definition 4.2. Let $(M, \omega, \Phi, \mathcal{T})$ and $(M', \omega', \Phi', \mathcal{T})$ be complexity one Hamiltonian T -manifolds. A **Φ -diffeomorphism** from M/T to M'/T is an orientation preserving diffeomorphism $f: M/T \rightarrow M'/T$ such that $\overline{\Phi'} \circ f = \overline{\Phi}$, and such that f and f^{-1} lift to Φ - T -diffeomorphisms in a neighbourhood of each exceptional orbit. Here, $\overline{\Phi}$ and $\overline{\Phi'}$ are induced by the moment maps.

We now state the main result of this section.

Proposition 4.3. *Let $\mathcal{T} \subset \mathfrak{t}^*$ be an open subset, $\Delta \subset \mathcal{T}$ a convex subset, and $\rho: \Delta \rightarrow \mathbb{R}_{>0}$ a function. Let \mathfrak{U} be a cover of \mathcal{T} by convex open sets. For each $U \in \mathfrak{U}$, let (M_U, ω_U, Φ_U) be a complexity one space over U with image $\Phi_U = U \cap \Delta$ and Duistermaat-Heckman function $\rho|_U$. For each U and V in \mathfrak{U} , let*

$$f_{UV}: M_V|_{U \cap V}/T \rightarrow M_U|_{U \cap V}/T$$

be a Φ -diffeomorphism, such that $f_{UV} \circ f_{VW} = f_{UW}$ on $M_W|_{U \cap V \cap W}/T$ for all $U, V, W \in \mathfrak{U}$. Then, after possibly passing to a refinement of the cover, there exist Φ - T -diffeomorphisms $g_{UV}: M_V|_{U \cap V} \rightarrow M_U|_{U \cap V}$ that lift f_{UV} and such that $g_{UV} \circ g_{VW} = g_{UW}$ on $M_W|_{U \cap V \cap W}$ for all $U, V, W \in \mathfrak{U}$.

⁵ This notion of a differential structure on quotient spaces was used by Schwarz [Sch]. An axiomatization of “differential structure” appeared in [Sik].

Under the assumptions of Proposition 4.3, let Q denote the topological space obtained from the disjoint union $\bigsqcup_{U \in \mathfrak{U}} M_U/T$ by identifying x with $f_{UV}(x)$ for all U and V in \mathfrak{U} and all $x \in M_V|_{U \cap V}/T$. Let

$$\overline{\Phi}: Q \rightarrow \mathcal{T}$$

denote the map induced by the moment maps. As in the proof of Proposition 3.1, Q is paracompact.

We define a differential structure on Q by declaring a real-valued function to be smooth if it lifts to a smooth function on each M_U ; notice that this is well defined. We can use smooth partitions of unity on the spaces M_U and \mathcal{T} to construct smooth partitions of unity on Q .

For any abelian Lie group A , let A^∞ denote the sheaf of smooth functions to A . Let $\mathcal{H}_{A^\infty}^i$ denote the presheaf on \mathcal{T} which associates the group $\check{H}^i(\overline{\Phi}^{-1}(U); A^\infty)$ to each open set $U \subset \mathcal{T}$. We will need the following lemma:

Lemma 4.4. *In the above situation,*

$$\check{H}^2(\mathcal{T}, \mathcal{H}_{T^\infty}^0) = 0.$$

Proof. Every short exact sequence of sheaves on Q gives rise to a long exact sequence in Čech cohomology. Therefore, the short exact sequence of sheaves on Q ,

$$0 \rightarrow \ell \rightarrow \mathfrak{t}^\infty \rightarrow T^\infty \rightarrow 1,$$

gives rise to a long exact sequence of presheaves on \mathcal{T}

$$(4.5) \quad 0 \rightarrow \mathcal{H}_\ell^0 \rightarrow \mathcal{H}_{\mathfrak{t}^\infty}^0 \rightarrow \mathcal{H}_{T^\infty}^0 \rightarrow \mathcal{H}_\ell^1 \rightarrow \mathcal{H}_{\mathfrak{t}^\infty}^1 \rightarrow \cdots.$$

Because the sheaf \mathfrak{t}^∞ is fine, $\check{H}^1(W, \mathfrak{t}^\infty) = 0$ for all open sets $W \subset Q$. Hence, $\mathcal{H}_{\mathfrak{t}^\infty}^1$ is the zero presheaf. Thus (4.5) breaks up into two short exact sequences of presheaves,

$$0 \rightarrow \mathcal{H}_\ell^0 \rightarrow \mathcal{H}_{\mathfrak{t}^\infty}^0 \rightarrow \kappa \rightarrow 0 \quad \text{and} \quad 0 \rightarrow \kappa \rightarrow \mathcal{H}_{T^\infty}^0 \rightarrow \mathcal{H}_\ell^1 \rightarrow 0,$$

where κ denotes the kernel of the homomorphism $\mathcal{H}_{T^\infty}^0 \rightarrow \mathcal{H}_\ell^1$. From these short exact sequences we get long exact sequences

$$(4.6) \quad \cdots \rightarrow \check{H}^i(\mathcal{T}, \mathcal{H}_{\mathfrak{t}^\infty}^0) \rightarrow \check{H}^i(\mathcal{T}, \kappa) \rightarrow \check{H}^{i+1}(\mathcal{T}, \mathcal{H}_\ell^0) \rightarrow \check{H}^{i+1}(\mathcal{T}, \mathcal{H}_{\mathfrak{t}^\infty}^0) \rightarrow \cdots$$

and

$$(4.7) \quad \cdots \rightarrow \check{H}^i(\mathcal{T}, \kappa) \rightarrow \check{H}^i(\mathcal{T}, \mathcal{H}_{T^\infty}^0) \rightarrow \check{H}^i(\mathcal{T}, \mathcal{H}_\ell^1) \rightarrow \cdots.$$

Because $\mathcal{H}_{\mathfrak{t}^\infty}^0(U) = \mathfrak{t}^\infty(\overline{\Phi}^{-1}(U))$ for all $U \subset \mathcal{T}$, the sheaf $\mathcal{H}_{\mathfrak{t}^\infty}^0$ is a fine sheaf, and so

$$\check{H}^i(\mathcal{T}, \mathcal{H}_{\mathfrak{t}^\infty}^0) = 0 \quad \text{for all} \quad i > 0.$$

Hence, (4.6) implies that $\check{H}^i(\mathcal{T}, \kappa) = \check{H}^{i+1}(\mathcal{T}, \mathcal{H}_\ell^0)$ for all $i > 0$. Thus, (4.7) becomes

$$\cdots \rightarrow \check{H}^{i+1}(\mathcal{T}, \mathcal{H}_\ell^0) \rightarrow \check{H}^i(\mathcal{T}, \mathcal{H}_{T^\infty}^0) \rightarrow \check{H}^i(\mathcal{T}, \mathcal{H}_\ell^1) \rightarrow \cdots$$

for all $i > 0$. The claim now follows immediately from Proposition 3.1. \square

The proof of Proposition 4.3 will use the following result.

Lemma 4.8. *Let (M, ω, Φ, U) and (M', ω', Φ', U) be complexity one spaces that have the same Duistermaat-Heckman function. Then every Φ -diffeomorphism $f: M/T \rightarrow M'/T$ lifts to a Φ - T -diffeomorphism from M to M' .*

Proof. Lemma 4.10 of [KT01] reads as follows:

Let Y be a local model for a non-exceptional orbit with a moment map $\Phi_Y: Y \rightarrow \mathfrak{t}^*$. Let W and W' be invariant open subsets of Y . Let $g: W/T \rightarrow W'/T$ be a diffeomorphism which preserves the moment map. Then g lifts to an equivariant diffeomorphism from W to W' .

Therefore, by Definition 4.2 and the local normal form theorem, every orbit in M/T has a neighbourhood on which f lifts to a Φ - T -diffeomorphism.

Condition (3.2) of [KT01] reads as follows:

- (*) The restriction map $H^2(M/T; \mathbb{Z}) \rightarrow H^2(\Phi^{-1}(y)/T; \mathbb{Z})$ is one-to-one for some regular value y of Φ .

Lemma 4.11 of [KT01] reads as follows:

Let (M, ω, Φ, U) and (M', ω', Φ', U) be complexity one spaces that satisfy Condition (*) and have the same Duistermaat-Heckman measure. Then every homeomorphism from M/T to M'/T that locally lifts to a Φ - T -diffeomorphism also lifts globally to a Φ - T -diffeomorphism from M to M' .

The lemma follows from this and Proposition 3.3. \square

We will also need the following result from [HaSa]:

Theorem 4.9 ([HaSa]). *Let a torus T act on a manifold M . Let $h: M \rightarrow M$ be an equivariant diffeomorphism that sends each orbit to itself. Then there exists a smooth invariant function $f: M \rightarrow T$ such that $h(m) = f(m) \cdot m$ for all $m \in M$.*

Proof of Proposition 4.3. Fix any U and V in \mathfrak{U} . Since U and V , and hence $U \cap V$, are convex, we can apply Lemma 4.8 to the spaces $M_U|_{U \cap V}$ and $M_V|_{U \cap V}$. Thus, there exists a Φ - T -diffeomorphism

$$F_{UV}: M_V|_{U \cap V} \rightarrow M_U|_{U \cap V}$$

that lifts f_{UV} .

For every $U, V, W \in \mathfrak{U}$, by Theorem 4.9, $F_{UV} \circ F_{VW} \circ F_{UW}^{-1}$ is given by acting by a smooth T -invariant function $M_U|_{U \cap V \cap W} \rightarrow T$. This function is the pull-back of a smooth function

$$(4.10) \quad h_{UVW}: Q|_{U \cap V \cap W} \rightarrow T.$$

On quadruple intersections, we have

$$(4.11) \quad (h_{UVW})(h_{UVX})^{-1}(h_{UWX})(h_{VWX})^{-1} = 1.$$

This is a cocycle condition; hence, the h_{UVW} represent a cohomology class in $H^2(\mathfrak{U}, \mathcal{H}_{T^\infty}^0)$. By Lemma 4.4, after possibly passing to a refinement of the cover \mathfrak{U} , there exist smooth T -invariant functions

$$B_{UV}: Q|_{U \cap V} \rightarrow T$$

such that

$$(4.12) \quad B_{UV} B_{VW} B_{UW}^{-1} = h_{UVW}$$

on triple intersections.

Then

$$g_{UV}(x) := (B_{UV}(x))^{-1} \cdot F_{UV}(x)$$

are liftings of the f_{UV} 's that satisfy the required compatibility condition. \square

5. GLUING SYMPLECTIC FORMS

The last ‘‘local to global’’ step is to modify the symplectic forms on the local pieces so that they agree on overlaps.

Proposition 5.1. *Let an $n-1$ dimensional torus T act on an oriented $2n$ dimensional manifold M , and let $\Phi: M \rightarrow \mathcal{T}$ be an invariant proper map to an open subset $\mathcal{T} \subset \mathfrak{t}^*$. Assume that $\Delta = \text{image } \Phi$ is convex. Fix a function $\rho: \Delta \rightarrow \mathbb{R}_{>0}$ and an open cover \mathfrak{U} of \mathcal{T} .*

Assume that, for all $U \in \mathfrak{U}$, there exists an invariant symplectic form ω_U on $\Phi^{-1}(U)$ with moment map $\Phi|_U$ and Duistermaat-Heckman function $\rho|_U$ such that ω_U is compatible with the given orientation. Then there exists an invariant symplectic form ω' on M with moment map Φ and Duistermaat-Heckman function ρ such that ω' is compatible with the given orientation.

Let a compact Lie group G act on a manifold M , and let $\{\xi_M\}_{\xi \in \mathfrak{g}}$ be the vector fields that generate this action. A differential form β on M is **basic** if it is G invariant and horizontal, that is, $\iota_{\xi_M}\beta = 0$ for all $\xi \in \mathfrak{g}$. The basic differential forms on M constitute a differential complex $\Omega_{\text{basic}}^*(M)$ whose cohomology coincides with the Čech cohomology of the topological quotient M/G ; see [Kos].

We will need the following technical lemma; cf. [KT01, Lemma 3.6].

Lemma 5.2. *Let an $(n - 1)$ dimensional abelian group T act faithfully on a $2n$ dimensional manifold M . Let $\Phi: M \rightarrow \mathfrak{t}^*$ be a smooth invariant map. Let ω_0 and ω_1 be invariant symplectic forms on M with moment map Φ that induce the same orientation on M . Let α be a basic two-form on M such that $\alpha(\xi, \eta) = 0$ for all $\xi, \eta \in \ker d\Phi$. Let λ_0 and λ_1 be non-negative functions on M such that $\lambda_0 + \lambda_1 = 1$. Then*

$$\lambda_0 \omega_0 + \lambda_1 \omega_1 + \alpha$$

is non-degenerate and induces the same orientation as ω_0 and ω_1 .

Proof. Let $x \in M$ be a point with stabilizer H ; let h be the dimension of H . By the local normal form theorem, a neighbourhood of the orbit of x with the symplectic form ω_0 is equivariantly symplectomorphic to a neighbourhood of the orbit $\{[t, 0, 0]\}$ in the model $T \times_H \mathbb{C}^{h+1} \times \mathfrak{h}^0$. The tangent space at x splits as $\mathfrak{t}/\mathfrak{h} \oplus \mathfrak{h}^0 \oplus \mathbb{C}^{h+1}$, where $\mathfrak{t}/\mathfrak{h}$ is the tangent space to the orbit. By the definition of the moment map, the forms $\omega_0|_x$ and $\omega_1|_x$ are given by block matrices of the form

$$\begin{pmatrix} 0 & I & 0 \\ -I & * & * \\ 0 & * & \tilde{\omega}_0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & I & 0 \\ -I & * & * \\ 0 & * & \tilde{\omega}_1 \end{pmatrix}$$

where I is the natural pairing between the vector space $\mathfrak{t}/\mathfrak{h}$ and its dual, \mathfrak{h}^0 , and where $\tilde{\omega}_0$ and $\tilde{\omega}_1$ are linear symplectic forms on \mathbb{C}^{h+1} with the same moment map and the same orientation. By our assumptions, $\alpha|_x$ is given by a block matrix of the form

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & * & * \\ 0 & * & 0 \end{pmatrix}.$$

Hence, $(\lambda_0 \omega_0 + \lambda_1 \omega_1 + \alpha)|_x$ is given by a block matrix of the form

$$\begin{pmatrix} 0 & I & 0 \\ -I & * & * \\ 0 & * & \tilde{\omega} \end{pmatrix}$$

where

$$\tilde{\omega} = \lambda_0(x)\tilde{\omega}_0 + \lambda_1(x)\tilde{\omega}_1.$$

It suffices to show that $\tilde{\omega}$ is non-degenerate and induces the same orientation as $\tilde{\omega}_0$ and $\tilde{\omega}_1$.

Case 1. Suppose that the stabilizer of x is trivial. Then $\tilde{\omega}_0$ and $\tilde{\omega}_1$ are non-zero two-forms on \mathbb{C} that induce the same orientation, and so $\tilde{\omega}$ is non-degenerate and induces the same orientation.

Case 2. Suppose that the stabilizer of x is non-trivial. Viewing $\tilde{\omega}$ as a translation invariant differential two-form on \mathbb{C}^{h+1} , it is enough to find some $v \in \mathbb{C}^{h+1}$ such that $\tilde{\omega}|_v$ is non-degenerate and induces the same orientation as $\tilde{\omega}_0|_v$ and $\tilde{\omega}_1|_v$. We choose $v \in \mathbb{C}^{h+1}$ whose stabilizer is trivial and apply Case 1 to the H action on \mathbb{C}^{h+1} . \square

Proof of Proposition 5.1. Given $j \in \mathbb{N}$, define a sheaf $\tilde{\Omega}_{\text{basic}}^j$ on \mathcal{T} by

$$\tilde{\Omega}_{\text{basic}}^j(U) = \Omega_{\text{basic}}^j(\Phi^{-1}(U)) \quad \text{for all open } U \subset \mathcal{T}.$$

Consider the double complex

$$K^{i,j} = \check{C}^i(\mathfrak{U}, \tilde{\Omega}_{\text{basic}}^j).$$

Let $d: K^{i,j} \rightarrow K^{i,j+1}$ denote the de-Rham differential, and let $\delta: K^{i,j} \rightarrow K^{i+1,j}$ denote the Čech differential.

To prove the proposition, it will be enough to find $\beta \in C^1(\mathfrak{U}, \tilde{\Omega}_{\text{basic}}^1)$ such that $\delta\beta = 0$ and $d\beta_{VW} = \omega_V - \omega_W$ for all V and W in \mathfrak{U} . To see this, let $\{\lambda_U\}_{U \in \mathfrak{U}}$ be the pull back to M of a smooth partition of unity on \mathcal{T} subordinate to \mathfrak{U} . Define

$$\omega'_V := \sum_{U \in \mathfrak{U}} \lambda_U \omega_U + \sum_{U \in \mathfrak{U}} d\lambda_U \wedge \beta_{UV} \in \Omega^2(\Phi^{-1}(V)) \quad \text{for all } V \in \mathfrak{U}.$$

Since $\delta\beta = 0$ and $\sum_{U \in \mathfrak{U}} \lambda_U = 1$,

$$\omega'_V - \omega'_W = \sum_{U \in \mathfrak{U}} d\lambda_U \wedge (\beta_{UV} - \beta_{UW}) = d\left(\sum_{U \in \mathfrak{U}} \lambda_U\right) \wedge \beta_{WV} = 0.$$

Therefore, the ω'_V glue together to give a global form $\omega' \in \Omega^2(M)$. Since each ω_U is an invariant symplectic form, and since $d\lambda_U(\xi) = 0$ for all $U \in \mathfrak{U}$ and all $\xi \in \ker d\Phi$, by repeated application of Lemma 5.2 ω' is non-degenerate and is compatible with the given orientation. Moreover,

$$\begin{aligned} \omega_V + \sum_{U \in \mathfrak{U}} d(\lambda_U \beta_{UV}) &= \omega_V + \sum_{U \in \mathfrak{U}} (d\lambda_U \wedge \beta_{UV} + \lambda_U (\omega_U - \omega_V)) \\ &= \omega_V - \sum_{U \in \mathfrak{U}} \lambda_U \omega_V + \omega'_V = \omega'_V. \end{aligned}$$

Thus, each ω_V and ω'_V differ by the exterior derivative of a basic one-form. This implies that ω' is closed, and so it is a symplectic form compatible with the given orientation. It also implies that ω' is invariant,

has the same moment map Φ as ω_V , and has the same Duistermaat-Heckman function ρ as ω_V .

As a first step towards finding the required cochain, we will show that we may assume that there exists $\beta \in C^1(\mathfrak{U}, \tilde{\Omega}_{\text{basic}}^1)$ such that $d\beta_{VW} = \omega_V - \omega_W$ for all V and W in \mathfrak{U} . After possibly passing to a refinement, we may assume that every $U \in \mathfrak{U}$ is convex. As we mentioned earlier, Condition (3.2) of [KT01] reads as follows:

- (*) The restriction map $H^2(M/T; \mathbb{Z}) \rightarrow H^2(\Phi^{-1}(y)/T; \mathbb{Z})$ is one-to-one for some regular value y of Φ .

Moreover, Lemma 3.5 of [KT01] reads as follows:

Let (M, ω, Φ, U) and (M', ω', Φ', U) be complexity one spaces that satisfy Condition (*) and have the same Duistermaat-Heckman measure. Then for every Φ - T -diffeomorphism $g: M \rightarrow M'$ there exists a basic one-form β on M such that $d\beta = g^*\omega' - \omega$.

Hence, the claim follows immediately from Proposition 3.3.

Next, we will show that we may assume that there exists $\gamma \in \check{C}^2(\mathfrak{U}, \tilde{\Omega}_{\text{basic}}^0)$ such that $\delta\gamma = 0$ and $\delta\beta = d\gamma$. For all $j \in \mathbb{N}$, define a presheaf $\mathcal{H}_{\mathbb{R}}^j$ on \mathcal{T} by $\mathcal{H}_{\mathbb{R}}^j(U) = \check{H}^j(\Phi^{-1}(U)/T; \mathbb{R})$ for all open $U \subset \mathcal{T}$. Recall that the Čech cohomology of $\Phi^{-1}(U)/T$ coincides with the cohomology of $(\Omega_{\text{basic}}^*(\Phi^{-1}(U)), d)$. Since $\delta^2\beta = 0$ and $d\delta\beta = 0$, the cochain $\delta\beta$ represents a cohomology class in $\check{H}^2(\mathfrak{U}; \mathcal{H}_{\mathbb{R}}^1)$. By Proposition 3.1, $\check{H}^2(\mathcal{T}, \mathcal{H}_{\mathbb{R}}^1) = 0$. Hence, after passing to a refinement, there exists $\beta' \in \check{C}^1(\mathfrak{U}, \tilde{\Omega}_{\text{basic}}^1)$ such that $d\beta' = 0$ and such that $\delta\beta$ and $\delta\beta'$ agree as elements of $\check{C}^2(\mathfrak{U}, \mathcal{H}_{\mathbb{R}}^1)$, i.e., there exists $\gamma \in \check{C}^2(\mathfrak{U}, \tilde{\Omega}_{\text{basic}}^0)$ such that $\delta\beta - \delta\beta' = d\gamma$. By replacing β by $\beta - \beta'$, we may assume that $\delta\beta = d\gamma$, as required. Since $\check{H}^3(\mathcal{T}, \mathcal{H}_{\mathbb{R}}^0) = 0$ by Proposition 3.1, we may assume that $\delta\gamma = 0$ by a similar argument.

Finally, we will use the fact that $\tilde{\Omega}_{\text{basic}}$ is a fine sheaf to show that we may assume that $\delta\beta = 0$, as required. Define $\eta \in \check{C}^1(\mathfrak{U}, \tilde{\Omega}_{\text{basic}}^0)$ by

$$\eta_{VW} = \sum_{U \in \mathfrak{U}} \lambda_U \gamma_{UVW} \quad \text{for all } V, W \in \mathfrak{U}.$$

Since $\delta\gamma = 0$, $\delta\eta = \gamma$, and so $\delta d\eta = d\gamma = \delta\beta$. Hence, we may replace β by $\beta - d\eta$. \square

6. RECONSTRUCTION

By breaking a space into the moment map preimages of small open subsets of \mathfrak{t}^* , and then gluing them back together, we obtain a special case of Theorem 1. This theorem is easier to prove than our other

existence theorems, in that it does not require the “local existence” results proved in Sections 7–9, i.e., it does not require us to determine which spaces can occur as preimages of small open subsets of \mathfrak{t}^* .

Theorem 6.1. *Let $(M, \omega, \Phi, \mathcal{T})$ be a tall complexity one space of genus g . Let Σ be a closed oriented surface of genus g , and let $f: M_{\text{exc}} \rightarrow \Sigma$ be any painting. Then there exists a tall complexity one space $(M', \omega', \Phi', \mathcal{T})$ with the same moment image and Duistermaat-Heckman function as M whose painting is equivalent to f .*

Since every tall complexity one space $(M, \omega, \Phi, \mathcal{T})$ has a convex moment image $\Delta = \Phi(M)$, a positive Duistermaat-Heckman function ρ , and a skeleton $S = M_{\text{exc}}$, Theorem 6.1 is simply the special case of Proposition 6.2 below with $\mathfrak{U} = \{\mathcal{T}\}$. (Proposition 6.2 is also a key ingredient in the proofs of Theorems 1, 2, and 3; see §10.)

Proposition 6.2. *Let \mathcal{T} be a convex open subset of \mathfrak{t}^* , $\Delta \subset \mathcal{T}$ a convex subset, and $\rho: \Delta \rightarrow \mathbb{R}_{>0}$ a positive function. Let (S, π) be a skeleton over \mathcal{T} , Σ a closed oriented surface of genus g , and $f: S \rightarrow \Sigma$ a painting. Finally, let \mathfrak{U} be a cover of \mathcal{T} by convex open sets.*

Suppose that for each $U \in \mathfrak{U}$ there exists a complexity one space (M_U, ω_U, Φ_U) of genus g over U with moment image $\Delta \cap U$ and Duistermaat-Heckman function $\rho|_{\Delta \cap U}$ whose skeleton is isomorphic to $S \cap \pi^{-1}(U)$. Then there exists a complexity one space $(M, \omega, \Phi, \mathcal{T})$ with moment image Δ and Duistermaat-Heckman function ρ whose painting is equivalent to f .

Proof. By Proposition 20.1 from the elephant [KT03] (see Proposition 6.3 below), after (possibly) passing to a refinement of \mathfrak{U} , there exists Φ -diffeomorphisms $h_{VU}: M_U/T|_{U \cap V} \rightarrow M_V/T|_{U \cap V}$ such that $h_{WV} \circ h_{VU} = h_{WU}$ on triple intersections and the following property holds.

If $(M, \omega, \Phi, \mathcal{T})$ is a complexity one space such that for every $U \in \mathfrak{U}$ there exists a Φ - T -diffeomorphism $\lambda_U: M|_U \rightarrow M_U$ so that h_{VU} is the map induced by the composition $\lambda_V \circ (\lambda_U)^{-1}$, then the painting associated to M is equivalent to f .

By Proposition 4.3, after passing to a refinement of \mathfrak{U} , there exist Φ - T -diffeomorphisms $g_{VU}: M_U|_{U \cap V} \rightarrow M_V|_{U \cap V}$ that lift h_{VU} and such that $g_{WV} \circ g_{VU} = g_{WU}$ on every triple intersection.

We use these Φ - T -diffeomorphisms to glue together the manifolds M_U . This gives an oriented $2n$ -dimensional manifold M with a T action and a T -invariant proper map $\Phi: M \rightarrow \mathcal{T}$. Moreover, there exists

a Φ - T -diffeomorphism $\lambda_U: M|_U \rightarrow M_U$ for each $U \in \mathfrak{U}$ such that $g_{VU} = \lambda_V \circ (\lambda_U)^{-1}$ on each double intersection.

For each $U \in \mathfrak{U}$, the pullback $\omega'_U := \lambda_U^* \omega_U$ is a T -invariant symplectic form on $\Phi^{-1}(U)$ with moment map $\Phi|_U$ and Duistermaat-Heckman function $\rho|_U$ and such that ω'_U is compatible with the given orientation. Therefore, by Proposition 5.1, there exists a T -invariant symplectic form ω on M with moment map Φ and Duistermaat-Heckman function ρ and such that ω is compatible with the given orientation. Finally, by the property above, the painting associated to $(M, \omega, \Phi, \mathcal{T})$ is equivalent to f . \square

For the reader's convenience, we now reformulate Proposition 20.1 from [KT03]:

Proposition 6.3. *Let \mathcal{T} be an open subset of \mathfrak{t}^* and $\Delta \subset \mathcal{T}$ a convex closed subset. Let $f: S \rightarrow \Sigma$ be a painting, where Σ is a closed oriented surface of genus g and (S, π) is a skeleton over \mathcal{T} . Let \mathfrak{U} be a cover of \mathcal{T} by convex open sets. For each $U \in \mathfrak{U}$, let $(M_U, \omega_U, \Phi_U, U)$ be a tall complexity one space of genus g over U , so that $\text{image } \Phi_U = U \cap \Delta$ and so that the set of exceptional orbits $(M_U)_{\text{exc}}$ is isomorphic to the restriction $S|_U := S \cap \pi^{-1}(U)$.*

Then, after possibly refining the open cover, one can associate to each U and V in \mathfrak{U} a Φ -diffeomorphism $h_{VU}: M_U/T|_{U \cap V} \rightarrow M_V/T|_{U \cap V}$ such that $h_{WV} \circ h_{VU} = h_{WU}$, and such that the following holds.

If $(M, \omega, \Phi, \mathcal{T})$ is a tall complexity one space such that for every $U \in \mathfrak{U}$ there exists a Φ - T -diffeomorphism $\lambda_U: M|_U \rightarrow M_U$ so that h_{VU} is the map induced by the composition $\lambda_V \circ (\lambda_U)^{-1}$, then the painting associated to M is equivalent to f .

PART II: CLASSIFICATION

7. COMPATIBILITY OF SKELETON

Let $(M, \omega, \Phi, \mathcal{T})$ be a tall complexity one space. The purpose of this section is to show that the set M_{exc} of exceptional orbits is a tall skeleton over \mathcal{T} , the moment image $\Phi(M)$ is a convex Delzant subset of \mathcal{T} , and the set $\Phi(M)$ and skeleton M_{exc} are compatible. See Definitions 1.13, 1.17, and 1.21.

We begin with an important observation.

Lemma 7.1. *Let $Y = T \times_H \mathbb{C}^{h+1} \times \mathfrak{h}^0$ be a complexity one model in which $\{[\lambda, 0, 0]\}$ is a non-exceptional orbit. Then, after possibly permuting the coordinates, $Y = T \times_H \mathbb{C}^h \times \mathbb{C} \times \mathfrak{h}^0$ and H acts on \mathbb{C}^h through an isomorphism with $(S^1)^h$. Consequently, every orbit in Y is non-exceptional, and the image of Y is a Delzant cone.*

Proof. Inside the model, the set of points that have stabilizer H and that lie in the same moment fiber as $[\lambda, 0, 0]$ is $T \times_H (\mathbb{C}^{h+1})^H \times \{0\}$, where $(\mathbb{C}^{h+1})^H$ is the subspace fixed by H . Since $\{[\lambda, 0, 0]\}$ is not an exceptional orbit, this subspace is not trivial. The result then follows from a dimension count. \square

We can now prove the main results of this section.

Lemma 7.2. *Let $(M, \omega, \Phi, \mathcal{T})$ be a tall complexity one space. The set M_{exc} of exceptional orbits, labelled by the isotropy representations in M and equipped with the map $\overline{\Phi}: M_{\text{exc}} \rightarrow \mathfrak{t}^*$ induced by the moment map, is a tall skeleton over \mathcal{T} .*

Proof. By construction, M_{exc} satisfies the local requirement in the definition of a tall skeleton. By the local normal form theorem and Lemma 7.1, the set of non-exceptional orbits is open, and hence M_{exc} is closed in M/T . So the restriction $\overline{\Phi}|_{M_{\text{exc}}}: M_{\text{exc}} \rightarrow \mathcal{T}$ is proper. \square

Lemma 7.3. *The moment image of a tall complexity one model is a Delzant cone.*

Proof. Let $Y = T \times_H \mathbb{C}^{h+1} \times \mathfrak{h}^0$ be a tall complexity one model with moment map Φ_Y , and let $\alpha = \Phi_Y([\lambda, 0, 0])$. By Lemma 2.2, there exists a non-exceptional orbit x in $\Phi_Y^{-1}(\alpha)$. Let Y_x be the corresponding complexity one model. By Lemma 7.1, $\text{image } \Phi_{Y_x}$ is a Delzant cone at α . By the local normal form theorem and the stability of the moment map for Y_x and for Y (see Section 2), there exists a neighbourhood U of α in \mathcal{T} such that $U \cap \text{image } \Phi_Y = U \cap \text{image } \Phi_{Y_x}$. Because $\text{image } \Phi_Y$ and $\text{image } \Phi_{Y_x}$ are invariant under dilations about α , this implies that they are equal. \square

Lemma 7.4. *Let $(M, \omega, \Phi, \mathcal{T})$ be a tall complexity one space. Then the moment image $\Phi(M)$ is a convex Delzant subset of \mathcal{T} that is compatible with the tall skeleton M_{exc} .*

Proof. Because \mathcal{T} is convex, M is connected, and $\Phi: M \rightarrow \mathcal{T}$ is proper, $\Phi(M)$ is a convex closed subset of \mathcal{T} ; see Section 2. Let x be a T -orbit in M , let Y be the corresponding model, and let $\alpha = \overline{\Phi}(x)$. By the local normal form theorem, stability for the moment map on Y , and stability of the moment map on M , there exists a neighbourhood U of α such that $\Phi(M) \cap U = \text{image } \Phi_Y \cap U$. The claim now follows from Corollary 2.5 and Lemma 7.3. \square

8. DUISTERMAAT-HECKMAN FUNCTIONS FOR TALL COMPLEXITY ONE MODELS

The purpose of this section is to define the Duistermaat-Heckman functions for truncations of a tall complexity one model (Definition 8.6) and to prove their basic properties (Corollaries 8.22 and 8.23). These functions were used in Section 1 to define *compatibility* of a (Duistermaat-Heckman) function and a tall skeleton; see Definition 1.23.

Lemma 8.1. *Let $Y = T \times_H \mathbb{C}^{h+1} \times \mathfrak{h}^0$ be a tall complexity one model. The torus*

$$G := T \times_H (S^1)^{h+1}$$

acts faithfully on this model. Let $i_T: T \rightarrow G$ denote the inclusion map. Then there exists a unique $(h+1)$ -tuple of non-negative integers (ξ_0, \dots, ξ_h) such that the following sequence is well defined and exact:

$$(8.2) \quad \{1\} \longrightarrow T \xrightarrow{i_T} G \xrightarrow{P} S^1 \longrightarrow \{1\},$$

$$\text{where } P([\lambda, a]) = a^\xi := \prod_{k=0}^h a_k^{\xi_k}.$$

The sequence

$$(8.3) \quad \{1\} \longrightarrow H \xrightarrow{\chi} (S^1)^{h+1} \xrightarrow{a \mapsto \prod_{k=0}^h a_k^{\xi_k}} S^1 \longrightarrow \{1\}$$

is also exact, where $\chi: H \rightarrow (S^1)^{h+1}$ is the embedding through which H acts on \mathbb{C}^{h+1} .

Proof. Lemma 8.1 follows from Lemmas 5.2, 5.3, and 5.8 of [KT01]. For completeness, we give a direct argument.

For any integers ξ_0, \dots, ξ_h , the sequence (8.2) is well defined and exact if and only if the sequence (8.3) is exact.

Because the quotient $(S^1)^{h+1}/\chi(H)$ is a one dimensional compact connected Lie group, there exist integers ξ_0, \dots, ξ_h such that (8.3) is exact; these integers are unique up to replacing (ξ_0, \dots, ξ_h) by $(-\xi_0, \dots, -\xi_h)$.

Let η_0, \dots, η_h be the weights for the H action on \mathbb{C}^{h+1} . Differentiating the relation $\chi(h)^\xi = 1$ gives $\sum_{k=0}^h \xi_k \eta_k = 0$.

The quadratic moment map for the H action on \mathbb{C}^{h+1} is given by $\Phi_H(z) = \sum_{k=0}^h \pi |z_k|^2 \eta_k$. Because Y is tall, the level set $\Phi_H^{-1}(0)$ contains more than one orbit. Hence, there exist complex numbers z_0, \dots, z_h , not all zero, such that $\sum_{k=0}^h \pi |z_k|^2 \eta_k = 0$.

Because the action is effective, the space of solutions (x_0, \dots, x_h) of the equation $\sum x_k \eta_k = 0$ is one dimensional. Hence, the previous two paragraphs imply that the vectors (ξ_0, \dots, ξ_h) and $(|z_0|^2, \dots, |z_h|^2)$ are

proportional. So, after possibly replacing (ξ_0, \dots, ξ_h) by $(-\xi_0, \dots, -\xi_h)$, the integers ξ_0, \dots, ξ_h are all non-negative. \square

Definition 8.4. We call the map $P: G \rightarrow S^1$ described above the **defining monomial**; cf. [KT01, Definition 5.12]. A **complementary circle** to T in G is a homomorphism $J: S^1 \rightarrow G$ such that $P \circ J = \text{id}_{S^1}$.

Let $\mathfrak{g}_{\mathbb{Z}}$ denote the integral lattice in \mathfrak{g} and $\mathfrak{g}_{\mathbb{Z}}^*$ the weight lattice in \mathfrak{g}^* . Thus,

$$\mathfrak{g}_{\mathbb{Z}} \cong \text{Hom}(S^1, G) \quad \text{and} \quad \mathfrak{g}_{\mathbb{Z}}^* \cong \text{Hom}(G, S^1).$$

Remark 8.5. Complementary circles always exist. To see this, note that the short exact sequence (8.2) gives rise to a short exact sequence of lattices, $\{0\} \rightarrow \ell \rightarrow \mathfrak{g}_{\mathbb{Z}} \rightarrow \mathbb{Z} \rightarrow \{0\}$. Any splitting of this sequence determines a complementary circle $J: S^1 \rightarrow G$ to T in G .

Let (M, ω, Φ) be a Hamiltonian T -manifold, and let $A \subset M$ be a measurable subset. The **Duistermaat-Heckman measure for the restriction of Φ to A** is the push-forward by the moment map of the restriction to A of the Liouville measure; a real valued function on $\Phi(A)$ is the **Duistermaat-Heckman function** for this restriction if its product with Lebesgue measure is the Duistermaat-Heckman measure. As before, it is almost unique; see the discussion on page 11.

Definition 8.6. Let $Y = T \times_H \mathbb{C}^{h+1} \times \mathfrak{h}^0$ be a tall complexity one model with moment map $\Phi_Y: Y \rightarrow \mathfrak{t}^*$. A real valued function ρ on a subset of \mathfrak{t}^* is the **Duistermaat-Heckman function for a truncation of the model** if there exist a complementary circle J to T in G and a positive number κ such that ρ is the Duistermaat-Heckman function for the restriction of Φ_Y to the subset

$$(8.7) \quad Y_{J,\kappa} := \varphi_J^{-1}((-\infty, \kappa])$$

of Y , where $\varphi_J: Y \rightarrow \mathbb{R}$ is the moment map for the resulting circle action on Y , normalized by $\varphi_J([\lambda, 0, 0]) = 0$.

Let Y be a tall complexity one space, and let $\alpha = \Phi_Y([\lambda, 0, 0])$. We will show in Corollaries 8.22 and 8.23 that there exist Duistermaat-Heckman functions for truncations of the model Y , that they are well defined and continuous on a neighbourhood of α in image Φ_Y , and that the difference of every two such functions is equal to an integral affine function on some neighbourhood of α in image Φ_Y .

Lemma 8.8. *Let $Y = T \times_H \mathbb{C}^{h+1} \times \mathfrak{h}^0$ be a tall complexity one model, and let $\tilde{\Phi}_Y: Y \rightarrow \mathfrak{g}^*$ be a moment map for the action of $G := T \times_H (S^1)^{h+1}$. There exists a linear isomorphism*

$$(8.9) \quad \mathfrak{g}^* \longrightarrow \mathfrak{h}^0 \times \mathbb{R}^{h+1}$$

with the following properties.

(1) The following diagram commutes:

$$\begin{array}{ccc} \mathfrak{g}^* & \xrightarrow{(8.9)} & \mathfrak{h}^0 \times \mathbb{R}^{h+1} \\ i_T^* \downarrow & & \downarrow (\nu, s) \mapsto (\chi^*(s), \nu) \\ \mathfrak{t}^* & \xrightarrow{\cong} & \mathfrak{h}^* \times \mathfrak{h}^0, \end{array}$$

where the bottom isomorphism is induced by the inner product on \mathfrak{t} that we have chosen, $i_T: T \rightarrow G$ is the inclusion map, and $\chi: H \rightarrow (S^1)^{h+1}$ is the embedding through which H acts on \mathbb{C}^{h+1} .

(2) The composition

$$Y \xrightarrow{\tilde{\Phi}_Y} \mathfrak{g}^* \xrightarrow{(8.9)} \mathfrak{h}^0 \times \mathbb{R}^{h+1}$$

has the form

$$(8.10) \quad [\lambda, z, \nu] \mapsto (\nu, \pi|z_0|^2, \dots, \pi|z_h|^2) + \text{constant}.$$

(3) Let ξ be the element of $\mathfrak{g}_{\mathbb{Z}}^*$ that corresponds to the defining monomial $P \in \text{Hom}(G, S^1)$, and let ξ_0, \dots, ξ_h be the exponents of the defining monomial. Then the isomorphism (8.9) carries ξ to $(0, (\xi_0, \dots, \xi_h))$.

Proof. Let $i_H: H \rightarrow T$ denote the inclusion map. The torus G is the quotient of $T \times (S^1)^{h+1}$ by the image of the H under embedding $a \mapsto (i_H(a)^{-1}, \chi(a))$. Hence,

$$(8.11) \quad \mathfrak{g}^* = \{(\gamma, s) \in \mathfrak{t}^* \times \mathbb{R}^{h+1} \mid i_H^*(\gamma) = \chi^*(s)\}.$$

Under the identification of \mathfrak{t}^* with $\mathfrak{h}^* \times \mathfrak{h}^0$, the space \mathfrak{g}^* becomes further identified with

$$(8.12) \quad \{(\beta, \nu, s) \in \mathfrak{h}^* \times \mathfrak{h}^0 \times \mathbb{R}^{h+1} \mid \beta = \chi^*(s)\}.$$

Now consider the composition

$$(8.13) \quad \mathfrak{g}^* \xrightarrow{\text{inclusion}} \mathfrak{t}^* \times \mathbb{R}^{h+1} \xrightarrow{\cong} \mathfrak{h}^* \times \mathfrak{h}^0 \times \mathbb{R}^{h+1} \xrightarrow{\text{projection}} \mathfrak{h}^0 \times \mathbb{R}^{h+1}.$$

The projection $(\beta, \nu, s) \mapsto (\nu, s)$ is a linear isomorphism from the space (8.12) – which is the image of \mathfrak{g}^* in $\mathfrak{h}^* \times \mathfrak{h}^0 \times \mathbb{R}^{h+1}$ – to $\mathfrak{h}^0 \times \mathbb{R}^{h+1}$. This proves that the composition (8.13) is a linear isomorphism.

Moreover, if (β, ν, s) is in (8.12) then $(\beta, \nu) = (\chi^*(s), \nu)$. This gives (1).

The moment map for the T action on Y , as a map to $\mathfrak{h}^* \times \mathfrak{h}^0$, has the form $[\lambda, z, \nu] \mapsto (\Phi_H(z), \nu) + \text{constant}$. The moment map for the $(S^1)^{h+1}$ action on Y has the form

$$[\lambda, z, \nu] \mapsto (\pi|z_0|^2, \dots, \pi|z_h|^2) + \text{constant}.$$

Therefore, the moment map for the G action on Y , as a map to $\mathfrak{h}^* \times \mathfrak{h}^0 \times \mathbb{R}^{h+1}$, has the form

$$[\lambda, z, \nu] \mapsto (\Phi_H(z), \nu, \pi|z_0|^2, \dots, \pi|z_h|^2) + \text{constant}.$$

This implies (2).

Since the restriction of $P \in \text{Hom}(G, S^1)$ to the subtorus T of G is trivial, and the restriction of P to the subtorus $(S^1)^{h+1}$ of G is the homomorphism $a \mapsto \prod_{k=0}^h a_k^{\xi_k}$, the natural embedding of \mathfrak{g}^* into $\mathfrak{t}^* \times \mathbb{R}^{h+1}$ carries ξ to $(0, (\xi_0, \dots, \xi_h))$. Projecting to $\mathfrak{h}^0 \times \mathbb{R}^{h+1}$, we get (3). \square

Lemma 8.14. *Let $Y = T \times_H \mathbb{C}^{h+1} \times \mathfrak{h}^0$ be a tall complexity one model with moment map $\Phi_Y: Y \rightarrow \mathfrak{t}^*$. Let $J \in \text{Hom}(S^1, G)$ be a complementary circle to T in $G := T \times_H (S^1)^{h+1}$, and let j be the corresponding element of $\mathfrak{g}_{\mathbb{Z}}$. Let $\tilde{\Phi}_Y: Y \rightarrow \mathfrak{g}^*$ be the unique G moment map that satisfies $i_T^* \circ \tilde{\Phi}_Y = \Phi_Y$ and $\langle \tilde{\Phi}_Y([\lambda, 0, 0]), j \rangle = 0$. There exists a unique continuous map*

$$\sigma: \text{image } \Phi_Y \rightarrow \mathfrak{g}^*$$

with the following properties.

- (1) $i_T^*(\sigma(\beta) + t\xi) = \beta$ for all $\beta \in \text{image } \Phi_Y$ and $t \in \mathbb{R}$.
- (2) $\text{image } \tilde{\Phi}_Y = \{\sigma(\beta) + t\xi \mid \beta \in \text{image } \Phi_Y \text{ and } t \geq 0\}$.
- (3) $\sigma(\tilde{\Phi}_Y([\lambda, 0, 0])) = \tilde{\Phi}_Y([\lambda, 0, 0])$.

Here, $i_T: T \rightarrow G$ is the inclusion map, and $\xi \in \mathfrak{g}_{\mathbb{Z}}^*$ corresponds to the defining monomial $P \in \text{Hom}(G, S^1)$.

Proof. Let $\chi: H \rightarrow (S^1)^{h+1}$ be the embedding through which H acts on \mathbb{C}^{h+1} . Let ξ_0, \dots, ξ_h be the (non-negative) exponents of the defining monomial. By (8.3), the level sets of the projection $\chi^*: \mathbb{R}^{h+1} \rightarrow \mathfrak{h}^*$ are the lines $s + \mathbb{R}(\xi_0, \dots, \xi_h)$.

After possibly reordering the coordinates, we may assume that

$$(8.15) \quad \xi_k > 0 \text{ for } 0 \leq k \leq h' \quad \text{and} \quad \xi_k = 0 \text{ for } h' < k \leq h;$$

let $h'' = h - h'$. Consider the subset $\partial \mathbb{R}_+^{h'+1} \times \mathbb{R}_+^{h''}$ of \mathbb{R}_+^{h+1} , consisting of $(h+1)$ -tuples of non-negative numbers in which at least one of the first $h' + 1$ entries is equal to zero. Consider the map

$$(8.16) \quad \partial \mathbb{R}_+^{h'+1} \times \mathbb{R}_+^{h''} \rightarrow \chi^*(\mathbb{R}_+^{h+1}), \quad s \mapsto \chi^*(s).$$

This map is a bijection because, by (8.15), the line $s + \mathbb{R}(\xi_0, \dots, \xi_h)$ meets $\partial\mathbb{R}_+^{h'+1} \times \mathbb{R}_+^{h''}$ exactly once for each $s \in \mathbb{R}_+^{h+1}$. Restricted to each closed facet, this map coincides with a linear isomorphism, and hence is open as a map to its image. It follows that the map (8.16) is open.

Since the map (8.16) is a homeomorphism, it has a continuous inverse

$$\sigma_H: \chi^*(\mathbb{R}_+^{h+1}) \rightarrow \mathbb{R}_+^{h+1}.$$

We claim that σ_H has the following properties.

- (1') $\chi^*(\sigma_H(\beta) + t(\xi_0, \dots, \xi_h)) = \beta$ for all $\beta \in \chi^*(\mathbb{R}_+^{h+1})$ and $t \in \mathbb{R}$.
- (2') $\mathbb{R}_+^{h+1} = \{\sigma_H(\beta) + t(\xi_0, \dots, \xi_h) \mid \beta \in \chi^*(\mathbb{R}_+^{h+1}) \text{ and } t \geq 0\}$.
- (3') $\sigma_H(0) = 0$.

Properties (1') and (3') follow immediately from the definition of σ_H . To prove (2'), consider $\beta \in \chi^*(\mathbb{R}_+^{h+1})$ and $t \in \mathbb{R}$. By (8.15), the fact that $\sigma_h(\beta) \in \partial\mathbb{R}_+^{h'+1} \times \mathbb{R}_+^{h''}$ implies that $\sigma_H(\beta) + t(\xi_0, \dots, \xi_h)$ lies in \mathbb{R}_+^{h+1} exactly if $t \geq 0$.

We may assume without loss of generality that $\Phi_Y([\lambda, 0, 0]) = 0$. Then the identification of \mathfrak{t}^* with $\mathfrak{h}^* \times \mathfrak{h}^0$ carries image Φ_Y onto $\chi^*(\mathbb{R}_+^{h+1}) \times \mathfrak{h}^0$. Define $\sigma: \text{image } \Phi_Y \rightarrow \mathfrak{g}^*$ so that the following diagram commutes:

$$\begin{array}{ccc} \text{image } \Phi_Y & \xrightarrow{\cong} & \chi^*(\mathbb{R}_+^{h+1}) \times \mathfrak{h}^0 \\ \sigma \downarrow & & \downarrow (\beta, \nu) \mapsto (\nu, \sigma_H(\beta)) \\ \mathfrak{g}^* & \xrightarrow{(8.9)} & \mathfrak{h}^0 \times \mathbb{R}^{h+1}, \end{array}$$

where (8.9) is the isomorphism defined in Lemma 8.8. By part (2) of that lemma, (8.9) carries image $\tilde{\Phi}_Y$ onto $\mathfrak{h}^0 \times \mathbb{R}_+^{h+1}$. Claims (1), (2), and (3) then follow from (1'), (2'), and (3'), respectively, by Parts (1) and (3) of Lemma 8.8. \square

A lattice element is *primitive* if it is not a multiple of another lattice element by an integer that is greater than one. The *rational length* of an interval $[x, y]$ with rational slope in \mathfrak{g}^* is equal to the positive number k such that $y - x = k\xi$ where ξ is a primitive lattice element.

Lemma 8.17. *Let $Y = T \times_H \mathbb{C}^{h+1} \times \mathfrak{h}^0$ be a tall complexity one model with moment map $\Phi_Y: Y \rightarrow \mathfrak{t}^*$. Let $J \in \text{Hom}(S^1, G)$ be a complementary circle to T in $G := T \times_H (S^1)^{h+1}$; let j be the corresponding element of $\mathfrak{g}_{\mathbb{Z}}$; and let $\varphi: Y \rightarrow \mathbb{R}$ be the moment map for the resulting circle action, normalized by $\varphi([\lambda, 0, 0]) = 0$. Then the $T \times S^1$ moment map $(\Phi_Y, \varphi): \Phi_Y \rightarrow \mathfrak{t}^* \times \mathbb{R}$ is proper, and each fiber contains at most one $T \times S^1$ orbit. Moreover, if $\sigma: \text{image } \Phi_Y \rightarrow \mathfrak{g}^*$ is the map given in*

Lemma 8.14, then

$$(8.18) \quad \text{image}(\Phi_Y, \varphi) = \{(\beta, s) \in \mathfrak{t}^* \times \mathbb{R} \mid \beta \in \text{image } \Phi_Y \text{ and } s \geq \langle \sigma(\beta), j \rangle\}.$$

Proof. Let $i_T: T \hookrightarrow G$ be the inclusion map. Let $\tilde{\Phi}_Y: Y \rightarrow \mathfrak{g}^*$ be a G moment map, normalized so that $(i_T^*, j) \circ \tilde{\Phi}_Y = (\Phi_Y, \varphi)$. Note that, in particular, $\langle \tilde{\Phi}_Y([\lambda, 0, 0], j) \rangle = \varphi([\lambda, 0, 0]) = 0$; cf. Lemma 8.14. By Lemma 8.8 (2), $\tilde{\Phi}_Y$ is proper and each fiber contains at most one G orbit.

Let $\xi \in \mathfrak{g}_{\mathbb{Z}}^*$ correspond to the defining monomial $P \in \text{Hom}(G, S^1)$. Because $P \circ J$ is the identity map, $\langle \xi, j \rangle = 1$. Moreover, since the homomorphism J splits the short exact sequence (8.2), the map $(i_T, J): T \times S^1 \rightarrow G$ is an isomorphism of groups. Hence, the induced map $(i_T^*, j): \mathfrak{g}^* \rightarrow \mathfrak{t}^* \times \mathbb{R}$ is a linear isomorphism. Therefore, by the first paragraph, (Φ_Y, φ) is proper and each fiber contains at most one $T \times S^1$ orbit.

Finally, (8.18) follows easily from properties (1) and (2) of Lemma 8.14. \square

Lemma 8.19. *Let $Y = T \times_H \mathbb{C}^{h+1} \times \mathfrak{h}^0$ be a tall complexity one model with moment map $\Phi_Y: Y \rightarrow \mathfrak{t}^*$. Let $J \in \text{Hom}(S^1, G)$ be a complementary circle to T in $G := T \times_H (S^1)^{h+1}$; let j be the corresponding element of $\mathfrak{g}_{\mathbb{Z}}$; and let $\varphi: Y \rightarrow \mathbb{R}$ be the moment map for the resulting circle action, normalized by $\varphi([\lambda, 0, 0]) = 0$. Given $\kappa \in \mathbb{R}$, define*

$$Y_{J, \kappa} = \varphi^{-1}((-\infty, \kappa]) \subset Y.$$

- (1) *If $\sigma: \text{image } \Phi_Y \rightarrow \mathfrak{g}^*$ is the map given in Lemma 8.14, then the function*

$$(8.20) \quad \beta \mapsto \kappa - \langle \sigma(\beta), j \rangle$$

from $\Phi_Y(Y_{J, \kappa})$ to \mathbb{R} is a Duistermaat-Heckman function for the restriction of Φ_Y to $Y_{J, \kappa}$.

- (2) *If $\kappa > 0$, then $\Phi_Y(Y_{J, \kappa})$ contains a neighbourhood of $\alpha = \Phi_Y([\lambda, 0, 0])$ in $\Phi_Y(Y)$.*

- (3) *The restriction of Φ_Y to $Y_{J, \kappa}$ is proper.*

Proof. Since by Lemma 8.17 each level set of (Φ_Y, φ) contains at most a single $T \times S^1$ orbit, the Duistermaat-Heckman measure for the restriction of (Φ_Y, φ) to $Y_{J, \kappa}$ is Lebesgue measure on the set $(\Phi_Y, \varphi)(Y_{J, \kappa})$. The Duistermaat-Heckman measure for the restriction of Φ_Y to $Y_{J, \kappa}$ is the push-forward of this measure under the projection map from $\mathfrak{t}^* \times \mathbb{R}$

to \mathfrak{t}^* . Finally, by (8.18),

$$(8.21) \quad (\Phi_Y, \varphi)(Y_{J,\kappa}) = \{(\beta, s) \in \mathfrak{t}^* \times \mathbb{R} \mid \beta \in \text{image } \Phi_Y \text{ and } \kappa \geq s \geq \langle \sigma(\beta), j \rangle\}.$$

Claim (1) follows immediately.

By Lemma 8.14, the function σ is continuous and $\langle \sigma(\alpha), j \rangle = \langle \tilde{\Phi}_Y([\lambda, 0, 0], j) = \varphi([\lambda, 0, 0]) = 0$. Therefore, if $\kappa > 0$, then the equation (8.21) implies that there is a neighbourhood U of α in \mathfrak{t}^* such that $\Phi_Y(Y) \cap U = \Phi_Y(Y_{J,\kappa}) \cap U$. This gives Claim (2).

Since σ is continuous, the equation (8.21) implies that the intersection $(K \times \mathbb{R}) \cap (\Phi_Y, \varphi)(Y_{J,\kappa})$ is compact for any compact set K . Moreover, by Lemma 8.17, (Φ_Y, φ) is proper. Claim (3) follows immediately. \square

Corollary 8.22. *For every tall complexity one model Y , there exist Duistermaat-Heckman functions for truncations of the model. Each such function is well defined and continuous on a neighbourhood of α in $\text{image } \Phi_Y$.*

Proof. By Remark 8.5, there exist complementary circles to T in G . The result then follows from Lemma 8.19 with any $\kappa > 0$. \square

Corollary 8.23. *Let $Y = T \times_H \mathbb{C}^{h+1} \times \mathfrak{h}^0$ be a tall complexity one model. Let ρ and ρ' be Duistermaat-Heckman functions for truncations of Y . Then the difference $\rho - \rho'$ is equal to an integral affine function on some neighbourhood of $\Phi_Y([\lambda, 0, 0])$ in $\text{image } \Phi_Y$.*

Proof. Let J and J' be complementary circles to T in $G := T \times_H (S^1)^{h+1}$, and let j and j' be the corresponding elements of $\mathfrak{g}_{\mathbb{Z}}$. Let φ and φ' be the associated moment maps, normalized by $\varphi([\lambda, 0, 0]) = \varphi'([\lambda, 0, 0]) = 0$. Let $P: G \rightarrow S^1$ be the defining monomial. Because $P \circ J = P \circ J' = \text{Id}_{S^1}$, there exists $\Delta j \in \mathfrak{t}_{\mathbb{Z}}$ such that

$$j - j' = i_T(\Delta j).$$

Let σ and σ' be the maps from $\text{image } \Phi_Y$ to \mathfrak{g}^* that are associated to J and J' , respectively, in Lemma 8.14. Let $\tilde{\Phi}_Y$ and $\tilde{\Phi}'_Y$ be the G moment maps, normalized as in Lemma 8.14. Because $i_T^* \circ \tilde{\Phi}_Y = i_T^* \circ \tilde{\Phi}'_Y$, there exists a real number c such that $\tilde{\Phi}_Y - \tilde{\Phi}'_Y = c\xi$, where $\xi \in \mathfrak{g}_{\mathbb{Z}}^*$ corresponds to $P \in \text{Hom}(P, S^1)$.

Therefore, parts (1) and (2) of Lemma 8.14 imply that

$$\sigma - \sigma' = c\xi.$$

Let κ and κ' be positive numbers. Let $\rho_{J,\kappa}$ and $\rho_{J',\kappa'}$ be the Duistermaat-Heckman functions for the restrictions of Φ_Y to $\varphi^{-1}((-\infty, \kappa])$ and

$\varphi'^{-1}((-\infty, \kappa'])$, respectively. By Lemma 8.19, there exists neighbourhood of $\Phi_Y([\lambda, 0, 0])$ in image Φ_Y where

$$\begin{aligned}
\rho_{J, \kappa}(\beta) - \rho_{J', \kappa'}(\beta) &= \kappa - \langle \sigma(\beta), j \rangle - \kappa' + \langle \sigma'(\beta), j' \rangle \\
&= \kappa - \langle \sigma(\beta), j \rangle - \kappa' + \langle \sigma(\beta) - c\xi, j - i_T(\Delta j) \rangle \\
&= \kappa - \kappa' - c - \langle \iota_T^*(\sigma(\beta)), \Delta j \rangle \\
&= \kappa - \kappa' - c - \langle \beta, \Delta j \rangle.
\end{aligned}$$

Here, the penultimate equality uses the fact that $\langle \xi, j \rangle = 1$, and the last equality follows from Lemma 8.14. \square

9. COMPATIBILITY OF THE DUISTERMAAT-HECKMAN FUNCTION, AND LOCAL EXISTENCE

This section achieves two goals. In Proposition 9.2 we prove that a Duistermaat-Heckman function of a complexity one space is compatible with its skeleton. In Proposition 9.10 we prove “local existence”: for any compatible values of our invariants, over sufficiently small open subsets of \mathfrak{t}^* there exists a tall complexity one space whose invariants take these values. The proofs of both propositions rely on a surgery that removes or adds exceptional orbits. In order to perform this surgery we identify a punctured neighbourhood of an exceptional orbit with a punctured neighbourhood of a non-exceptional orbit. This identification is done in Lemma 9.1.

We begin by setting up the relevant notation. Let C be a Delzant cone in \mathfrak{t}^* , and let (M_C, ω_C, Φ_C) be a symplectic toric manifold whose moment image is C . We recall how to obtain such a manifold. By Definition 1.17 there exist an integer $0 \leq k \leq n$ and a linear isomorphism $A: \mathbb{R}^n \rightarrow \mathfrak{t}^*$ that sends \mathbb{Z}^n onto the weight lattice ℓ^* such that $C = \alpha + A(\mathbb{R}_+^k \times \mathbb{R}^{n-k})$. We may take M_C to be the manifold

$$M_C := \mathbb{C}^k \times (T^*S^1)^{n-k},$$

with the standard symplectic structure; with the T -action given by the isomorphism $T \rightarrow (S^1)^k \times (S^1)^{n-k}$ induced by $A^*: \mathfrak{t} \rightarrow \mathbb{R}^n$; and with the moment map $\Phi_C(z, a, \eta) = \alpha + A(\pi|z_1|^2, \dots, \pi|z_k|^2, \eta_1, \dots, \eta_{n-k})$, where $z = (z_1, \dots, z_k) \in \mathbb{C}^k$ and $(a, \eta) \in (S^1)^{n-k} \times \mathbb{R}^{n-k} \cong (T^*S^1)^{n-k}$.

We also consider the manifold $M_C \times \mathbb{C}$ with the product symplectic structure. This manifold admits a T action on the first factor with moment map $(m, z) \mapsto \Phi_C(m)$ for all $m \in M_C$ and $z \in \mathbb{C}$. It also

admits a toric action of $T \times S^1$ with moment map $(m, z) \mapsto (\Phi_C(m), \kappa + \pi|z|^2)$, for any $\kappa \in \mathbb{R}$.

Given $\epsilon > 0$, let D_ϵ be the disk

$$D_\epsilon = \{z \in \mathbb{C} \mid \pi|z|^2 < \epsilon\}.$$

Lemma 9.1. *Let $Y = T \times_H \mathbb{C}^{h+1} \times \mathfrak{h}^0$ be a tall complexity one model with moment map $\Phi_Y: Y \rightarrow \mathfrak{t}^*$. Let J be a complementary circle to T in $G := T \times_H (S^1)^{h+1}$; and let $\varphi: Y \rightarrow \mathbb{R}$ be the moment map for the resulting circle action, normalized by $\varphi([\lambda, 0, 0]) = 0$. Let $\alpha = \Phi_Y([\lambda, 0, 0])$.*

- (1) *Let V be a neighbourhood of the orbit $\{[\lambda, 0, 0]\}$ in Y . Then there exists a neighbourhood U of α in \mathfrak{t}^* and a positive number κ' such that the preimage $(\Phi_Y, \varphi)^{-1}(U \times (-\infty, \kappa'))$ is contained in V .*
- (2) *Let (M_C, ω_C, Φ_C) be a symplectic toric manifold whose moment image is⁶ $C := \text{image } \Phi_Y$. For every positive number κ , there exist $\epsilon > 0$, a neighbourhood U of $\alpha \in \mathfrak{t}^*$, and a T equivariant symplectomorphism between*

$$\begin{aligned} \Phi_Y^{-1}(U) \cap \varphi^{-1}((\kappa, \kappa + \epsilon)) &\subset Y \quad \text{and} \\ \Phi_C^{-1}(U) \times (D_\epsilon \setminus \{0\}) &\subset M_C \times \mathbb{C} \end{aligned}$$

that intertwines $\varphi: Y \rightarrow \mathbb{R}$ and the map $(m, z) \mapsto \kappa + \pi|z|^2$. Here, T acts only on the first factor of $M_C \times \mathbb{C}$.

Proof. Let j be the element of $\mathbb{Z}_G \subset \mathfrak{g}$ that corresponds to $J \in \text{Hom}(S^1, G)$. Let $\sigma: \text{image } \Phi_Y \rightarrow \mathfrak{g}^*$ be given as in Lemma 8.14. Then by Lemma 8.14 and Lemma 8.17:

- (i) the map σ is continuous and $\langle \sigma(\alpha), j \rangle = 0$;
- (ii) $(\Phi_Y, \varphi): Y \rightarrow \mathfrak{t}^* \times \mathbb{R}$ is proper and each fiber contains at most one orbit; and
- (iii) $\text{image } (\Phi_Y, \varphi) = \{(\beta, s) \in \mathfrak{t}^* \times \mathbb{R} \mid \beta \in \Phi_Y(Y) \text{ \& } s \geq \langle \sigma(\beta), j \rangle\}$.

Let V be a neighbourhood of the orbit $\{[\lambda, 0, 0]\}$ in Y . By (ii) above, there exist a neighbourhood U of α in \mathfrak{t}^* and a positive number κ' such that the pre-image $(\Phi_Y, \varphi)^{-1}(U \times (-\kappa', \kappa'))$ is contained in V . Therefore, by (i) and (iii) above, after possibly shrinking U , the pre-image $(\Phi_Y, \varphi)^{-1}(U \times (-\infty, -\kappa'])$ is empty; this proves Claim (1).

Moreover, if κ is a positive number, then by (i) and (iii) above, there exists a neighbourhood W of $(\alpha, \kappa) \in \mathfrak{t}^* \times \mathbb{R}$ such that $\text{image}(\Phi_Y, \varphi) \cap W = (C \times \mathbb{R}) \cap W$. Consider Y and $M_C \times S^1 \times \mathbb{R}$ as symplectic

⁶The moment cone is a Delzant cone by Lemma 7.3.

manifolds with toric $T \times S^1$ actions and with moment maps (Φ_Y, φ) and $(\Phi_C, \kappa + \text{proj}_{\mathbb{R}})$, respectively. By the argument above, their moment images coincide with $C \times \mathbb{R}$, hence with each other, on a neighbourhood of (α, κ) . Because (ii) holds, the local normal form for toric actions (see [Del]) implies that, after possibly shrinking this neighbourhood, its preimages in Y and in $M_C \times S^1 \times \mathbb{R}$ are isomorphic. Thus, after possibly shrinking U , for sufficiently small ϵ the subset $\Phi_Y^{-1}(U) \cap \varphi^{-1}((\kappa - \epsilon, \kappa + \epsilon))$ of Y is isomorphic to the subset $\Phi_C^{-1}(U) \times S^1 \times (\kappa - \epsilon, \kappa + \epsilon)$ of $M_C \times S^1 \times \mathbb{R}$. Restricting to the preimage of $U \times (\kappa, \kappa + \epsilon)$, and composing further with the map $re^{i\theta} \mapsto (e^{i\theta}, \kappa + \pi r^2)$, which is an S^1 -equivariant symplectomorphism from $D_\epsilon \setminus \{0\}$ onto $S^1 \times (\kappa, \kappa + \epsilon)$ that carries the map $z \mapsto \kappa + \pi|z|^2$ to the map $(\lambda, s) \mapsto s$, we get Claim (2). \square

We are now ready to prove the first main result of this section.

Proposition 9.2. *Let $(M, \omega, \Phi, \mathcal{T})$ be a tall complexity one space. The skeleton and Duistermaat-Heckman function of M are compatible.*

Proof. Fix a point α in the moment image $\Delta = \Phi(M)$. By Lemma 7.2 and Corollary 2.6, there are only finitely many exceptional orbits x in $\Phi^{-1}(\alpha)$. For each such x , let $Y_x = T \times_{H_x} \times \mathbb{C}^{h_x+1} \times \mathfrak{h}_x^0$ be the tall complexity one model with moment map $\Phi_x: Y_x \rightarrow \mathfrak{t}^*$ corresponding to x .

By applying surgery to neighbourhoods of the exceptional orbits, we will construct a complexity one space (M', ω', Φ', U) with moment image $\Delta \cap U$ that has no exceptional orbits, where $U \subset \mathcal{T}$ is a convex open neighbourhood of α . Additionally, the Duistermaat-Heckman function $\rho': \Delta \cap U \rightarrow \mathbb{R}_{>0}$ of M' will satisfy

$$(9.3) \quad \rho' = \rho - \sum_x \rho_x,$$

where ρ_x is the Duistermaat-Heckman function for a truncation of the model Y_x for each exceptional orbit x in $\Phi^{-1}(\alpha)$, and the sum is over all such orbits. Because M' is a complexity one space with no exceptional orbits, ρ' is integral affine on $\Delta \cap U$. The proposition will follow immediately.

By Lemma 7.4 there exists a Delzant cone C at α and a convex open neighbourhood U of α in \mathfrak{t}^* such that $\Delta \cap U = C \cap U$. Let (M_C, ω_C, Φ_C) be a toric manifold with moment image C .

By the local normal form theorem, for each exceptional orbit x in $\Phi^{-1}(\alpha)$ there exists an isomorphism Ψ_x from an invariant neighbourhood V_x of the orbit $\{[\lambda, 0, 0]\}$ in Y_x to an invariant open subset of M

that carries $\{[\lambda, 0, 0]\}$ to x . Moreover, we may assume that the closures in M of the open subsets $\Psi_x(V_x)$ are disjoint.

Given an exceptional orbit x in $\Phi^{-1}(\alpha)$, let J_x be a complementary circle to T in $T \times_{H_x} (S^1)^{h_x+1}$; see Remark 8.5. Let φ_x be a moment map for the resulting circle action, normalized by $\varphi_x([\lambda, 0, 0]) = 0$. By Lemma 7.4, $\text{image } \Phi_x = C$. By Lemma 9.1, for sufficiently small $\epsilon > 0$, after possibly shrinking U , there exists $\kappa_x > 0$ such that $\Phi_x^{-1}(U) \cap \varphi_x^{-1}((-\infty, \kappa_x + \epsilon))$ is contained in V_x ; moreover, there exists an isomorphism between

$$(9.4) \quad \Phi^{-1}(U) \cap \Psi_x(\varphi_x^{-1}((\kappa_x, \kappa_x + \epsilon))) \subset M \quad \text{and}$$

$$(9.5) \quad \Phi_C^{-1}(U) \times (D_\epsilon \setminus \{0\}) \subset M_C \times \mathbb{C}$$

that intertwines $\varphi_x \circ \Psi_x^{-1}: \Psi_x(V_x) \rightarrow \mathbb{R}$ and the map $(m, z) \mapsto \kappa_x + \pi|z|^2$.

We construct M' by gluing together the spaces

$$(9.6) \quad \Phi^{-1}(U) \setminus \bigsqcup_x \Psi_x(\varphi_x^{-1}((-\infty, \kappa_x])) \subset M \quad \text{and}$$

$$(9.7) \quad \bigsqcup_x \Phi_C^{-1}(U) \times D_\epsilon \subset M_C \times \mathbb{C}$$

by identifying their isomorphic open subsets (9.4) and (9.5) for every exceptional orbit x in $\Phi^{-1}(\alpha)$.

The space M' is the union two *closed* subspaces: the images in M' of

$$(9.8) \quad \Phi^{-1}(U) \setminus \bigsqcup_x \Psi_x(\varphi_x^{-1}((-\infty, \kappa_x + \tfrac{1}{2}\epsilon))) \subset M \quad \text{and}$$

$$(9.9) \quad \bigsqcup_x \Phi_C^{-1}(U) \times \overline{D}_{\frac{1}{2}\epsilon} \subset M_C \times \mathbb{C},$$

where $\overline{D}_{\frac{1}{2}\epsilon} \subset \mathbb{C}$ is the closed disk. Because each of these is Hausdorff, M' is Hausdorff.

By construction, M' is a manifold with a T action, a symplectic form ω' , and a moment map Φ' . Moreover, as maps to U , the restriction of Φ to (9.8) and the restriction of Φ_C to (9.9) are both proper, and so Φ' is proper as well.

This yields a complexity one space (M', ω', Φ', U) with moment image $\Delta \cap U$ that has no exceptional orbits in $\Phi'^{-1}(\alpha)$. By Lemma 7.2, the restriction of Φ' to the set of exceptional orbits is proper. Therefore, after possibly shrinking U further, we may assume that M' has no exceptional orbits.

By Lemma 8.19, after possibly shrinking U , there is a well-defined Duistermaat-Heckman function $\rho_x: \Delta \cap U \rightarrow \mathbb{R}_{>0}$ for the restriction of Φ_x to $\varphi_x^{-1}((-\infty, \kappa_x])$, and thus for the restriction of Φ to $\Psi_x(\varphi_x^{-1}((-\infty, \kappa_x])$. Since M' and (9.6) differ by a set of measure zero, and since ρ_x is a Duistermaat-Heckman function for a truncation of the model Y_x , the Duistermaat-Heckman function ρ' for (M', ω', Φ') satisfies (9.3), as required. \square

We proceed to the second main result of this section:

Proposition 9.10 (Local existence). *Let (S, π) be a tall skeleton over an open subset $\mathcal{T} \subset \mathfrak{t}^*$, let $\Delta \subset \mathcal{T}$ be a convex Delzant subset that is compatible with (S, π) , and let g be a non-negative integer.*

- (1) *For any $\alpha \in \mathcal{T}$ there exists a convex open neighbourhood $U \subset \mathcal{T}$ of α and a tall complexity one space of genus g over U with moment image $\Delta \cap U$ whose skeleton is isomorphic to $S|_U$.*
- (2) *Let $\rho: \Delta \rightarrow \mathbb{R}_{>0}$ be a function that is compatible with (S, π) . Then for any $\alpha \in \mathcal{T}$ there exists a convex open neighbourhood $U \subset \mathcal{T}$ of α and a tall complexity one space of genus g over U with moment image $\Delta \cap U$, whose skeleton is isomorphic to $S|_U$ and whose Duistermaat-Heckman function is $\rho|_{\Delta \cap U}$.*

Proof of part (1). Let α be a point in \mathcal{T} and fix an arbitrary positive number b . By Corollary 2.6, the level set $\pi^{-1}(\alpha)$ in S is finite. We will choose a convex open neighbourhood $U \subset \mathcal{T}$ of α and construct a complexity one space (M', ω', Φ', U) of genus g with moment image $\Delta \cap U$ whose skeleton is isomorphic to $S|_U$. Additionally, the Duistermaat-Heckman function ρ' of M' will be smaller than b near α , and will satisfy

$$(9.11) \quad \rho' = c + \sum_{s \in \pi^{-1}(\alpha)} \rho_s,$$

where c is a positive real number and where each ρ_s is the Duistermaat-Heckman function for a truncation of the model Y_s associated to $s \in S$.

Choose a closed symplectic 2-manifold (Σ, η) of genus g and a positive number κ_s for each $s \in \pi^{-1}(\alpha)$ such that

$$\int_{\Sigma} \eta + \sum_{s \in \pi^{-1}(\alpha)} \kappa_s < b.$$

Since Δ is a Delzant subset, there exists a convex open neighbourhood U of α and a Delzant cone C at α such that $\Delta \cap U = C \cap U$. Let (M_C, ω_C, Φ_C) be a toric manifold with moment image C .

By the Darboux theorem, if $\epsilon > 0$ is sufficiently small, for each $x \in \pi^{-1}(\alpha)$ we can choose an open set W_s in Σ and a symplectomorphism $\Psi_s: D_\epsilon \rightarrow W_s$; let $x_s = \Psi_s(0)$. Furthermore, we may assume that the closures of the sets W_s are disjoint. Clearly, for each $s \in \pi^{-1}(\alpha)$, there is an isomorphism between

$$(9.12) \quad \Phi_C^{-1}(U) \times (D_\epsilon \setminus \{0\}) \subset M_C \times \mathbb{C} \quad \text{and}$$

$$(9.13) \quad \Phi_C^{-1}(U) \times (W_s \setminus \{x_s\}) \subset M_C \times \Sigma,$$

given by $(m, z) \mapsto (m, \Psi_s(z))$.

Let $Y_s = T \times_{H_s} \mathbb{C}^{h_s+1} \times \mathfrak{h}_s^0$ be the tall complexity one model with moment map $\Phi_s: Y_s \rightarrow \mathfrak{t}^*$ corresponding to $s \in \pi^{-1}(\alpha)$. Let J_s be a complementary circle to T in $T \times_{H_s} (S^1)^{h_s+1}$; see Remark 8.5. Let φ_s be a moment map for the resulting circle action, normalized by $\varphi_s([\lambda, 0, 0]) = 0$. Since (S, π) and Δ are compatible, image $\Phi_{Y_s} = C$. By part (2) of Lemma 9.1, after possibly shrinking U and ϵ , there exists an isomorphism between

$$(9.14) \quad \Phi_s^{-1}(U) \cap \varphi_s^{-1}((\kappa_s, \kappa_s + \epsilon)) \subset Y_s \quad \text{and}$$

$$(9.15) \quad \Phi_C^{-1}(U) \times (D_\epsilon \setminus \{0\}) \subset M_C \times \mathbb{C}$$

that intertwines $\varphi_s: Y_s \rightarrow \mathbb{R}$ and the map $(m, z) \mapsto \kappa_s + \pi|z|^2$.

Because the sets (9.13) and (9.14) are both isomorphic to the same set, there exists an isomorphism between them that intertwines the map $(m, \Psi_s(z)) \mapsto \kappa_s + \pi|z|^2$ and the map φ_s . We construct M' by gluing together the spaces

$$(9.16) \quad \Phi_C^{-1}(U) \times (\Sigma \setminus \{x_s\}_{s \in \pi^{-1}(\alpha)}) \subset M_C \times \Sigma \quad \text{and}$$

$$(9.17) \quad \bigsqcup_{s \in \pi^{-1}(\alpha)} \Phi_s^{-1}(U) \cap \varphi_s^{-1}((-\infty, \kappa_s + \epsilon)) \subset \bigsqcup_{s \in \pi^{-1}(\alpha)} Y_s$$

by identifying their isomorphic open subsets (9.13) and (9.14) for every $s \in \pi^{-1}(\alpha)$.

The space M' is the union of two *closed* subspaces: the images in M' of

$$(9.18) \quad \Phi_C^{-1}(U) \times \left(\Sigma \setminus \bigsqcup_{s \in \pi^{-1}(\alpha)} \Psi_s(D_{\epsilon/2}) \right) \subset M_C \times \Sigma \quad \text{and}$$

$$(9.19) \quad \bigsqcup_{s \in \pi^{-1}(\alpha)} \Phi_s^{-1}(U) \cap \varphi_s^{-1}((-\infty, \kappa_s + \epsilon/2]) \subset \bigsqcup_{s \in \pi^{-1}(\alpha)} Y_s.$$

Because each of these is Hausdorff, M' is Hausdorff.

By construction, M' is a manifold with a T action, a symplectic form ω' , and a moment map Φ' . Moreover, as a map to U , the restriction of

Φ_C to (9.18) is proper, and the restriction of each Φ_s to (9.19) is proper by part (3) of Lemma 8.19. Therefore, Φ' is proper as a map to U .

This yields a complexity one space (M', ω', Φ', U) with moment image $\Delta \cap U$, and an isomorphism from M'_{exc} onto a neighbourhood of $\pi^{-1}(\alpha)$ in S . Because π is proper, after possibly shrinking U further, we may assume that the image of M'_{exc} in S is $S|_U$. Since it is straightforward to check that M' has genus g , it remains to show that the Duistermaat-Heckman function ρ' of M' is smaller than b near α .

M' can be written as the disjoint union of (9.16) and

$$\bigsqcup_{s \in \pi^{-1}(\alpha)} \Phi_s^{-1}(U) \cap \varphi_s^{-1}((-\infty, \kappa_s]) \subset \bigsqcup_{s \in \pi^{-1}(\alpha)} Y_s.$$

Clearly, the function that takes the constant value $c = \int_{\Sigma} \eta$ on $C \cap U$ is a Duistermaat-Heckman function for the set (9.16). Moreover, by Lemma 8.19, after possibly shrinking U , for each $s \in \pi^{-1}(\alpha)$, the function $\rho_s: \Delta \cap U \rightarrow \mathbb{R}$ given by $\rho_s(\alpha) = \kappa_s - \langle \sigma(\alpha), j \rangle$ is a Duistermaat-Heckman function for the restriction of Φ_s to $\Phi_s^{-1}(U) \cap \varphi_s^{-1}((-\infty, \kappa_s])$. In particular, it is a Duistermaat-Heckman function for a truncation of the model Y_s . Thus, the Duistermaat-Heckman function ρ' for M' satisfies (9.11).

Since $\rho_s(\alpha) = \kappa_s - \langle \sigma(\alpha), j \rangle = \kappa_s$, we have chosen the positive numbers η and κ_s so that ρ' is smaller than b near α . \square

Proof of part (2). By part (1) there exists a complexity one space (M', ω', Φ', U) of genus g with moment image $\Delta \cap U$ whose skeleton is isomorphic to $S|_U$ such that the Duistermaat-Heckman function ρ' of M' satisfies $\rho'(\alpha) < \rho(\alpha)$ and (9.11). Thus, since ρ is compatible with (S, π) , Corollary 8.23 implies that $\rho - \rho'$ is integral affine on a neighbourhood of α in Δ . (Alternatively, this follows from Proposition 9.2.) Thus, since $\rho' < \rho$ near α , there exists $\kappa > 0$ and $\zeta \in \ell \subset \mathfrak{t}$ such that $\rho(\beta) - \rho'(\beta) = \kappa - \langle \beta - \alpha, \zeta \rangle$.

The tall complexity one Hamiltonian T -manifold $Y = M_C \times \mathbb{C}$ with moment map $\Phi_Y(m, z) = \Phi_C(m)$ is isomorphic to the tall complexity one model $T \times_H (\mathbb{C}^k \times \mathbb{C}) \times \mathfrak{h}^0$, where $T \cong (S^1)^n$, where $H \cong \{1\}^{n-k} \times (S^1)^k$, and where $\mathfrak{h}^0 \cong \mathbb{R}^{n-k}$. So we can apply Lemma 8.19 to it. We identify the corresponding torus $G = T \times_H (S^1)^{k+1}$ with $T \times S^1$, and we identify $[\lambda, 0, 0]$ with the point $(m_0, 0)$ such that $\Phi_C(m_0) = \alpha$.

Define $J: S^1 \rightarrow T \times S^1$ by $J(\lambda) = (\lambda^\zeta, \lambda)$. Let $\varphi: Y \rightarrow \mathbb{R}$ be the moment map for the resulting circle action, normalized by $\varphi(m_0, 0) = 0$. The $T \times S^1$ moment map $\tilde{\Phi}_Y: Y \rightarrow \mathfrak{t}^* \times \mathbb{R}$ given by $\tilde{\Phi}_Y(m, z) = (\Phi_C(m), \pi|z|^2 - \langle \alpha, \zeta \rangle)$ is normalized as in Lemma 8.14. Its moment image is $\tilde{\Phi}_Y(Y) = C \times [-\langle \alpha, \zeta \rangle, \infty)$. Hence, the map $\sigma: C \rightarrow \mathfrak{t}^* \times$

\mathbb{R} described in Lemma 8.14 is $\sigma(\beta) = (\beta, -\langle \alpha, \zeta \rangle)$. Therefore, by Lemma 8.19, the restriction of Φ_Y to $\varphi^{-1}((-\infty, \kappa])$ is proper and the Duistermaat-Heckman function for this restriction is $\rho_{J, \kappa}(\beta) = \kappa - \langle \beta - \alpha, \zeta \rangle$.

By part (2) of Lemma 9.1, there exists a T equivariant symplectomorphism from $\Phi_Y^{-1}(U) \cap \varphi^{-1}((\kappa, \kappa + \epsilon))$ to $\Phi_C^{-1}(U) \times (D_\epsilon \setminus \{0\})$ that carries the map φ to the map $(m, z) \mapsto \kappa + \pi|z|^2$. If we glue this local model into M' following the same procedure as explained in the above proof of part (1), we get a new complexity one space, which satisfies all our requirements. \square

10. PROOF OF THE EXISTENCE THEOREMS

We are now ready to prove the existence theorems that we stated in Section 1.

Proof of Theorem 3. By part (2) of Proposition 9.10 for each point in \mathcal{T} there exists a convex open subset $U \subset \mathcal{T}$ containing the point and there exists a complexity one space of genus $g := \text{genus}(\Sigma)$ over U whose moment image is $\Delta \cap U$, whose skeleton is isomorphic to $S|_U$, and whose Duistermaat-Heckman function is $\rho|_U$. The result now follows from Proposition 6.2. \square

Proof of Theorem 1. This theorem is an immediate consequence of Lemma 7.4, Proposition 9.2, and Theorem 3. \square

Proof of Theorem 2. This theorem is an immediate consequence of Lemma 10.1 below and Theorem 3. \square

Lemma 10.1. *Let $\mathcal{T} \subset \mathfrak{t}^*$ be an open subset. Let $\Delta \subset \mathcal{T}$ be a convex Delzant subset. Let (S, π) be a skeleton over \mathcal{T} . If Δ and (S, π) are compatible, then there exists a function $\rho: \Delta \rightarrow \mathbb{R}_{>0}$ that is compatible with (S, π) .*

Proof. By part 1 of Proposition 9.10, there exists a cover \mathfrak{U} of \mathcal{T} by convex sets and, for each $U \in \mathfrak{U}$, a complexity one space $(M_U, \omega_U, \Phi_U, U)$ whose moment image is $U \cap \Delta$ and whose skeleton is $S|_U$. Let

$$\rho_U: U \cap \Delta \rightarrow \mathbb{R}$$

be its Duistermaat-Heckman function. By Proposition 9.2, ρ_U is compatible with the moment image $\Delta \cap U$ and the skeleton $(S|_U, \pi_U)$. Hence, by Definition 1.23, and Corollary 8.23, on every intersection the difference

$$\rho_U|_{U \cap V} - \rho_V|_{U \cap V}$$

is locally an integral affine function. Hence this difference is given by a locally constant function

$$h_{UV}: U \cap V \rightarrow \ell \oplus \mathbb{R}.$$

Let \mathcal{A} denote the sheaf of locally constant functions to $\ell \oplus \mathbb{R}$. Because Δ is convex, the Čech cohomology $H^1(\Delta, \mathcal{A})$ is trivial. Hence, after possibly passing to a refinement of the cover, there exist locally constant functions

$$(10.2) \quad h_U: U \rightarrow \ell \oplus \mathbb{R}$$

such that $h_U - h_V = h_{UV}$ on $U \cap V$. Therefore, we can define $\rho: \Delta \rightarrow \mathbb{R}$ by $\rho|_U = \rho_U - h_U$ for all $U \in \mathfrak{U}$, where h_U also denotes the integral affine function given by (10.2). \square

REFERENCES

- [AhHa] Kazushi Ahara and Akio Hattori, *4 dimensional symplectic S^1 -manifolds admitting moment map*, J. Fac. Sci. Univ. Tokyo Sect. IA, Math. **38** (1991), 251–298.
- [AlH] Klaus Altmann and Jürgen Hausen, *Polyhedral divisors and algebraic torus actions*, Mathematische Annalen **334** (2006), 557–607.
- [AlHS] Klaus Altmann, Jürgen Hausen, and Hendrik Süß, *Gluing affine torus actions via divisorial fans*, Transformation Groups **13**, no. 2 (2008), 215–242.
- [AlP] Klaus Altmann and Lars Petersen, *Cox rings of rational complexity one T -varieties*, arXiv:1009.0478 [math:AG].
- [Ati] Michael Atiyah, *Convexity and commuting hamiltonians*, Bull. London Math. Soc. **14** (1982), 1–15.
- [Au90] Michèle Audin, *Hamiltoniens périodiques sur les variétés symplectiques compactes de dimension 4*, Géométrie symplectique et mécanique, Proceedings 1988, C. Albert ed., Springer Lecture Notes in Math. **1416** (1990).
- [Au91] Michèle Audin, *The topology of torus actions on symplectic manifolds*, Progress in Mathematics **93**, Birkhuser Verlag, Basel, 1991.
- [BjKa] Christina Bjorndahl and Yael Karshon, *Revisiting Tietze-Nakajima: local and global convexity for maps*, Canadian Journal of Mathematics **62** (2010), no. 5, 975–993.
- [Chi] River Chiang, *Complexity one Hamiltonian $SU(2)$ and $SO(3)$ actions*, Amer. J. Math. **127** (2005), no. 1, 129–168.
- [Del] Thomas Delzant, *Hamiltoniens périodiques et image convexe de l’application moment*, Bull. Soc. Math. France **116** (1988), 315–339.
- [God] Roger Godement, *Topologie algébrique et théorie des faisceaux*, Hermann, Paris, 1964.
- [Gon] Eduardo Gonzalez, *Classifying semi-free Hamiltonian S^1 -manifolds*, Int. Math. Res. Not. IMRN **2011**, no. 2, 387–418.
- [GLS] Victor Guillemin, Eugene Lerman, and Shlomo Sternberg, *Symplectic Fibrations and Multiplicity Diagrams*, Cambridge Univ. Press, 1996.

- [GS82] Victor Guillemin and Shlomo Sternberg, *Convexity properties of the moment mapping*, Invent. Math. **67** (1982), 491–513.
- [GS84] Victor Guillemin and Shlomo Sternberg, *A normal form for the moment map*, in: Differential geometric methods in mathematical physics (Jerusalem, 1982), 161–175, Math. Phys. Stud., 6, Reidel, Dordrecht, 1984.
- [HaSa] André Haeffliger and Éliane Salem, *Actions of tori on orbifolds*, Ann. Global Anal. Geom. **9** (1991), 37–59.
- [Igl] Patrick Iglésias, *Les $SO(3)$ -variétés symplectiques et leur classification en dimension 4*, Bull. Soc. Math. France **119** (1991), 371–396.
- [Kar] Yael Karshon, *Periodic Hamiltonian flows on four dimensional manifolds*, Mem. Amer. Math. Soc. **672** (1999).
- [KL] Yael Karshon and Eugene Lerman, *Non-compact symplectic toric manifolds*, arXiv:0907.2891 [math.SG].
- [KT01] Yael Karshon and Susan Tolman, *Centered complexity one Hamiltonian torus actions*, Trans. Amer. Math. Soc. **353** (2001), 4831–4861.
- [KT03] Yael Karshon and Susan Tolman, *Complete invariants for Hamiltonian torus actions with two dimensional quotients*, Journal of Symplectic Geometry 2 no.1, (2003), 25–82.
- [KT05] Yael Karshon and Susan Tolman, *The Gromov width of complex Grassmannians*, Algebr. Geom. Topol. **5** (2005), 911–922.
- [KZ] Yael Karshon and Fabian Ziltener, *Hamiltonian group actions on exact symplectic manifolds with proper momentum maps are standard* (tentative title), in preparation.
- [KKMS] George Kempf, Finn F. Knudsen, David Mumford, and Berhard Saint-Donat, *Toroidal Embeddings I*, Lecture Notes in Math. **339**, Springer-Verlag, Berlin - New York, 1973.
- [Kim] Min Kyu Kim, *Frankel’s theorem in the symplectic category*, Trans. Amer. Math. Soc. **358** (2006), no. 10, 4367–4377.
- [Kn11] Friedrich Knop, *Automorphisms of multiplicity free Hamiltonian manifolds*, J. Amer. Math. Soc. **24** (2011), no. 2, 567–601.
- [Kos] Jean-Louis Koszul, *Sur certains groupes de transformations de Lie*, Colloque international du Centre National de la Recherche Scientifique **52** (1953), 137–141.
- [LMTW] Eugene Lerman, Eckhard Meinrenken, Susan Tolman, and Chris Woodward, *Nonabelian convexity by symplectic cuts*, Topology **37** (1998), 245–259.
- [LeTo] Eugene Lerman and Susan Tolman, *Hamiltonian torus actions on symplectic orbifolds and toric varieties*, Trans. Amer. Math. Soc., **349** (1997), no. 10, 4201–4230.
- [Li03] Hui Li, *Semi-free Hamiltonian circle actions on 6-dimensional symplectic manifolds*, Trans. Amer. Math. Soc. **355** (2003), no. 11, 4543–4568.
- [Li05] Hui Li, *On the construction of certain 6-dimensional symplectic manifolds with Hamiltonian circle actions*, Trans. Amer. Math. Soc. **357** (2005), no. 3, 983–998.
- [LiTo] Hui Li and Susan Tolman, *Hamiltonian circle actions with minimal fixed sets*, arXiv:0905.4049 [math.SG].

- [Los] Ivan Losev, *Proof of the Knop conjecture*, Ann. Inst. Fourier (Grenoble) **59** (2009), 1105–1134.
- [Mar] Charles-Michel Marle, *Modèle d'action hamiltonienne d'un groupe de Lie sur une variété symplectique*, *Rendiconti del Seminario Matematico* **43** (1985), 227–251, Università Politecnica, Torino.
- [McD] Dusa McDuff, *Some 6-dimensional Hamiltonian S^1 -manifolds*. J. Topol. **2** (2009), no. 3, 589–623.
- [Mor] D. Morton, *Hamiltonian GKM manifolds with Low Betti Numbers*, PhD. thesis, University of Illinois at Urbana Champaign, 2011.
- [Mos] Jürgen Moser, *On the volume elements on a manifold*, Trans. Amer. Math. Soc. **120** (1965), 286–294.
- [Nak] S. Nakajima, *Über konvexe Kurven and Fläschen*, Tohoku Mathematical Journal, voll. **29** (1928), 227–230.
- [Noz] Hiraku Nozawa, *Five dimensional K-contact manifolds of rank 2*, arXiv:0907.0208 [math.SG].
- [Sch] Gerald W. Schwarz, *Lifting smooth homotopies of orbit spaces*, I.H.E.S. Publ. Math. **51** (1980), 37–135.
- [Sik] Roman Sikorski, *Wstęp do geometrii różniczkowej*. (Polish) [Introduction to differential geometry] Biblioteka Matematyczna, Tom 42. [Mathematics Library, Vol. 42] Pastwowe Wydawnictwo Naukowe, Warsaw, 1972.
- [Sja] Reyner Sjamaar, *Convexity properties of the moment mapping re-examined*, Adv. Math. **138** (1998), 46–91.
- [Tie] Heinrich Tietze, *Über Konvexität im kleinen und im großen und über gewisse den Punkten einer Menge zugeordnete Dimensionszahlen*. (German) Math. Z. **28** (1928), no. 1, 697–707.
- [Tim96] Dmitry A. Timashëv, *G-manifolds of complexity 1*, (Russian) Uspekhi Mat. Nauk **51** (1996), no. 3 (309), 213–214; English translation: Russian Math. Surveys **51** (1996), 567–568.
- [Tim97] Dmitry A. Timashëv, *Classification of G-manifolds of complexity 1*, (Russian) Izv. Ross. Akad. Nauk Ser. Mat. **61** (1997), 127–162; English translation: Izv. Math. **61** (1997), 363–397.
- [Tol98] Susan Tolman, *Examples of non-Kähler Hamiltonian torus actions*, Invent. Math. **131** (1998), no. 2, 299–310.
- [Tol10] Susan Tolman, *On a symplectic generalization of Petrie's conjecture*, Trans. Amer. Math. Soc. **362** (2010), 3963–3996.
- [Vol] Robert Vollmert, *Toroidal embeddings and polyhedral divisors*, Int. J. Algebra **4** (2010), no. 5–8, 383–388.
- [Woo] Chris Woodward, *The classification of transversal multiplicity-free group actions*. Ann. Global Anal. Geom. **14** (1996), no. 1, 3–42.

UNIVERSITY OF TORONTO, 40 ST. GEORGE STREET, TORONTO ONTARIO M5S 2E4, CANADA

E-mail address: karshon@math.toronto.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN, URBANA, IL 61801

E-mail address: stolman@math.uiuc.edu